Abstract The new spinor-unit field representation of the electromagnetism [21] (with quark and lepton sources) is integrated via minimal coupling with standard Einstein gravitation, to formulate a Lagrangian model of the very early universe. A completely new solution to the coupled Einstein-Maxwell equations, with sources, is derived. These equations are generalized somewhat, but not in a way that violates any physical principles.

The solution of the coupled Euler-Lagrange field equations yields a scale factor $a(t)$ (comoving coordinates) that initially exponentially increases N e-folds from $a(0) \approx 0$ to $a_1 = a(0)e^N$ ($N = 60$ is illustrated), then exponentially decreases, then exponentially increases to a_1, and so on almost periodically. (Oscillatory cosmological models are not knew, and have been derived from string theory and loop quantum gravity.) It is not known if the scale factor escapes this periodic trap.

This model is noteworthy in several respects: {1} All fundamental fields other than gravity are realized by spinor fields. {2} A plausible connection between the *unit* field **u** and the generalization of the photon wave function with a form of Dark Energy is described, and a simple natural scenario is outlined that allocates a fraction of the total energy of the Universe to this form of Dark Energy. {3} A solution of an analog of the pure Einstein-Maxwell equations is found using an approach that is in marked contrast with the method followed to obtain a solution of the well known Friedmann model of a radiation-dominated universe.

Spinor-Unit Field Representation of Electromagnetism Applied to a Model Inflationary Cosmology

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1 Introduction

1.1 Description of inflationary cosmology

Inflationary cosmology [10,14,1] is a widely accepted and largely successful theory of the physics of the very early universe. Recent observational data from important experiments such as WMAP [8,7,13] and the Sloan Digital Sky Survey [25,9,24], and others support [8,7,27] the main predictions of inflation theory. Inflation employs a period of accelerated expansion at the beginning of the Universe to solve the monopole, horizon and flatness problems. The simplest models of inflation are based on the slow-roll of a scalar inflaton field. The simple slow-roll inflaton potential model does not have the "exit from inflation" issue of the basic cosmological constant scenario, since the period of cosmic acceleration is followed by a radiation-dominated era, and then by a transient matter dominated phase [23]. After some mean time period, the inflaton excites Standard Model degrees of freedom and the hot big bang commences. Quantum fluctuations during the early time evolution of the inflaton field generate primordial curvature fluctuation spectra [17] that evolve into macroscopic perturbations whose time evolution leads to cosmological structure formation. Most models predict a scale-invariant spectra of energy density perturbations and gravitational waves that act as seeds for structure formation. Moreover, the power spectrum of curvature fluctuations at horizon crossing during inflation may be related to the angular power spectrum of cosmic microwave background (CMB) fluctuations at recombination, and yield predictions of the anisotropies in the cosmic microwave background radiation.

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Yet, to date, no inflaton-type field model has gained universal acceptance. In this paper we study a new cosmological model based on the coupled {1} Einstein, {2} electromagnetic (with quark and lepton sources) and the {3} *unit* field **u** equations that exhibits an oscillatory form of inflation/deflation. This model may also provide some insight into the nature of Dark-Energy. The photon field is represented according to [21] using a real bosonic spinor field, and gives rise to the exact conventional interacting local $U(1)-$invariant Maxwell equations after inflation has ended. "Maxwell" refers to the electromagnetic sector of the Standard Model. For simplicity, we consider only one generation of leptons and quarks. Also for simplicity, this model employs a flat $(k = 0)$ universe Friedmann-Lemaître-Robertson-Walker (FLRW) line element with scale factor $a = a(t)$ (comoving coordinates) for the Einstein sector. However, we do not expect our qualitative results to change for non-zero, constant k, or with the inclusion of additional Standard Model degrees of freedom.

The solution of the coupled field equations yields a scale factor $a(t) \equiv a[F(t)]$, where $F(t)$ is defined in Eq.[1] below, that initially exponentially increases N e-folds from $a(0) \approx 0$ to $a_1 = a(0)e^N$ ($N = 60$ is illustrated; N is strictly a function of initial conditions), then exponentially decreases to $a(0)$, then decreases to 0, then exponentially increases to a_1, then exponentially decreases to 0, and so on almost periodically. The energy density ϱ, discussed below, determines the depth of the potential well of the periodic trap. It is not known if the scale factor escapes this periodic trap. The exponential change is with respect to F. The exact dependence is given by Eq.[2] and is only approximately exponential for large values of $|F|$. Nevertheless, for brevity we use the descriptor "exponential". With respect to comoving time t the change is accelerated. The scale factor makes the transition from $a(0)$ to a_{max} in a finite comoving time that can be made as small as one chooses by changing initial conditions. Also, a_{max} can be made as large as one wants and $a(0)$ as small by changing initial conditions.

Central to the new cosmological model is the unit field **u** [21], which is a fundamental spinor field realization of the unit element of the split octonion algebra. ***Throughout this paper we work with the components u^a of u, rather than with u, to avoid the complexities of the non-associativity of the split octonion algebra.*** With respect to a canonical spinor basis of the algebra, its eight components $\mathbf{u} \leftrightarrow u^a$, $a = 1, ..., 8$, comprise a fundamental spinor that transform under a basic spinor representation of $\overline{SO(4,4)}$. In this paper the unit field is a section $u^a = u^a(x^\alpha)$ $\alpha = 1, ..., 4$ of a certain spacetime fiber bundle whose technical specification is given in Appendix 1. The unit field may be employed to construct a local $U(1)-$invariant covariant derivative, enabling a basic (bosonic) spinor field representation of the electromagnetic sector of the Standard Model [21].

In this paper the unit field is parameterized as

$$u^a = \frac{1}{\sqrt{2}} \left(0, e^{-\frac{F(t)}{2}}, 0, 0, 0, e^{\frac{F(t)}{2}}, 0, 0 \right), \tag{1}$$

where $F(t) \equiv 0$ characterizes the Minkowski spacetime of Einstein's special relativity (but not uniquely). This Lagrangian model predicts that

$$a(t)^3 \cosh{[F(t)]} = a(0)^3 \cosh{[F(0)]} = \text{constant}, \qquad (2)$$

i.e., $a(t)^3 \cosh{[F(t)]}$ is an exact integral of the motion. Since $F(t) \equiv 0$ characterizes the end of inflation, $\frac{a(t_{\text{end of inflation}})}{a(0)} = \cosh{[F(0)]}^{1/3}$, where $\cosh{[F(t)]} = \widetilde{\mathbf{u}}\mathbf{u} = [u^2(t)]^2 + [u^6(t)]^2$, is solely determined by the initial value $F(0)$. (Here, the tilde denotes the transpose.)

1.1.1 Radiation

The Segre classification of the energy-momentum tensor [and, in virtue of the Einstein field equations, of the Ricci tensor] of a pure radiation field, which is a null Maxwell field, is the algebraic type A3[(11,2)], with eigenvalue zero. Therefore the "radiation" that is modeled in the radiation-dominated Friedmann equations cannot be a solution of the Einstein-Maxwell equations because the Friedmann model stress-energy tensor is of type A1[(111),1], which is that of a non-tilted massive perfect fluid with timelike four-velocity. Type A1[(111),1] is also different from the types A1[(111,1)] (Λ cosmology), and A1[(11)(1,1)] (non-null Maxwell field). In the Friedmann universe radiation is an incoherent superposition of waves with random phases and polarizations. If one integrates the stress-energy tensor $T_{\mu\nu}$ of this electromagnetic radiation over solid angle then one finds that the average $\langle T_{\mu\nu}\rangle$ is of type A1[(111),1]. As an example, define the lightlike Minkowski vector $\ell_\alpha = (\sin(\theta)\cos(\varphi), \sin(\theta)\sin(\varphi), \cos(\theta), 1)$ and use the definition of $d\,T_{\mu\nu}$ given on page 521 of [28], which depends on the product $\ell^\alpha \ell_\beta$. Then $\frac{1}{4\pi}\int_0^{2\pi}\int_0^\pi \ell^\alpha \ell_\beta \sin(\theta)\,d\theta d\varphi = \text{diag}\,\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -1\right)$. $\langle T_{\mu\nu}\rangle$ defines/depends-on the radiation energy density $\varrho(t)$ and pressure $p(t)$ functions, and appears in the Einstein equations $G_{\mu\nu} = -2\langle T_{\mu\nu}\rangle$ that yield the radiation-dominated Friedmann equations. The equation of state relating the radiation pressure p and energy density ϱ, after the averaging has been performed, is $p = \frac{1}{3}\varrho$.

However, in this paper we solve an analog of the pure Einstein-Maxwell equations. We present a self-consistent model of gravity and radiation, with massive and electrically charged sources, that introduces the energy density $\varrho(t)$ and pressure $p(t)$ [26] functions in order to make the total energy momentum tensor homogeneous and isotropic (see Eq.[12]). There is no *a priori* equation of state relating $p(t)$ and $\varrho(t)$ because $p(t)$ is determined dynamically from the field equations (of course, $\dot\varrho(t)$ is always determined from the field equations).

It should emphasized that we do not "second quantize" the quantum fields of massive, electrically charged fermions. A field equation for a massive and electrically charged fermion, information-equivalent to the usual Dirac equation, is defined in terms of a real, eight component spinor field. The charge operator is not diagonal in this representation, as is well known from Schwinger's

work with real eight component Fermion spinor fields [22]. But for simplicity, the real eight component Fermion spinor fields are not "second quantized." This simplifying assumption may be justified by the fact that we study a temperature regime far above the electron (resp., down quark) rest mass; therefore electrons and positrons (resp., quarks and anti-quarks) never annihilate efficiently.

This paper is organized as follows: in the next section we define our notation and conventions; technical details are mostly relegated to the Appendices. In Section 3 the assumption that the universe is created in a non-equilibrium state is discussed, which is argued to imply that in order to make the total energy momentum tensor homogeneous and isotropic an additional term must be incorporated into the Lagrangian. Next the Lagrangian for this problem is defined. The Euler-Lagrange field equations are discussed in Section 5. A representative numerical approximation of the solutions of the field equations for the FLRW scale factor $a(t)$, for the split octonion unit spinor field $u^a(t)$ and for the electromagnetic split octonion spinor fields are presented. Lastly, we summarize our results.

2 Notation and Conventions

Let G denote the Newtonian gravitational constant, c the speed of light in vacuum; $\sqrt{\frac{\hbar G}{c^3}}$ is the Planck length ℓ. We employ units in which $4\pi G = 1 = \hbar = c = \epsilon_0 = \mu_0$. The elementary quantum of electric charge is $e = \sqrt{4\pi\alpha} \approx 0.30282212...$, where α is the fine-structure constant. The Planck time $t_P \approx 0.282095$. For simplicity we assume that all Standard Model charges other than electric are zero, and neglect all contributions to energy, momentum and charge from electrically charged gauge bosons. Also, in this paper "leptons and quarks" means only electrons, and up and down quarks, and their anti-particles.

Let $(\mathbb{X}_{3,1}; g)$ be a spacetime. $\mathbb{X}_{3,1}$ is a four-dimensional, Hausdorff, smooth manifold, and g is, almost everywhere [because the scale factor $a(t)$ may periodically vanish], a smooth pseudo-Riemannian metric [a locally Lorentz metric of signature $(+ + + \text{-})$] that is defined on $\mathbb{X}_{3,1}$. The manifold $\mathbb{X}_{3,1}$ and the metric g are assumed smooth (C^∞). We also require that $\mathbb{X}_{3,1}$ have a spin structure. We use the usual component notation in local charts. The tensor covariant derivative with respect to the symmetric connection associated with the metric g will be denoted by a semicolon. A partial derivative is sometimes denoted by a comma.

In our model both the unit field and the photon field are realized as smooth sections of a spacetime spinor bundle. Technical details are discussed in the Appendices, and may be briefly summarized as follows: the *unit* field **u**, has 8 real-valued field components $\mathbf{u} \leftrightarrow u^a = u^a(x^\alpha), a = 1, ...8, \ \alpha = 1, 2, 3, 4$, with respect to an oriented spinor basis of the split octonion algebra, that transform under the action of $\mathrm{Spin}(4, 4, \mathbb{R}) \equiv \overline{SO(4, 4; \mathbb{R})}$ as a basic type-1 8-component real-valued $\overline{SO(4, 4; \mathbb{R})}$ *bosonic* spinor field on $\mathbb{X}_{3,1}$. The photon

field $\Psi_2 \leftrightarrow \psi_2{}^a$ has 8 real-valued field components that transform as a basic type-2 8-component real-valued $\mathrm{Spin}(4, 4, \mathbb{R})$ **bosonic** spinor field on $\mathbb{X}_{3,1}$, $\psi_2^a = \psi_2^a(x^\alpha)$. [Upon restriction to $\overline{SO(3, 4; \mathbb{R})}$ the u^a and ψ_2^a transform in the same way.] Following Dirac, $\widetilde{\psi}$ denotes the transpose of a (real) 8-component spinor ψ.

The new definition of the photon wave function is based on a new local $U(1)$-invariant Lagrangian formulation of Maxwell's theory [21] that is cast in terms of the unit field $\mathbf{u}$ and ψ_2^a, and which revises the definition of the photon wave function, but whose associated Euler-Lagrange equations imply the conventional Maxwell equations exactly, in the flat spacetime limit. It cannot be overemphasized that in this formalism the electromagnetic interaction is carried by a real, 8-component, bosonic type-2 spinor field $\Psi_2 \leftrightarrow \psi_2{}^a$, which defines the four-vector potential A_α in the standard Maxwell theory via an interaction with the unit field, $\mathbb{A}^\alpha = \frac{1}{\sqrt{\widetilde{\mathbf{u}} \,\sigma \mathbf{u}}} \widetilde{\mathbf{u}} \,\sigma \,\overline{\tau}^\alpha \,\Psi_2$; the $\overline{\tau}^\alpha$ matrices are defined in the Appendix in Eq.[54]. In addition, the association to electric and magnetic fields $(\mathbf{E}, \mathbf{B})$ employs a Lagrange multiplier $\Psi_1 \leftrightarrow \psi_1{}^a$ [which is a real, 8-component, (bosonic) type-1 spinor field], related to the conventional $(\mathbf{E}, \mathbf{B})$ Minkowski frame components according to

$$
\psi_1{}^1 = \frac{(-B_{(3)} + B_{(4)}) \cosh\left[\frac{F(t)}{2}\right]}{\sqrt{2}}; \; \psi_1{}^2 = \frac{e^{-\frac{F(t)}{2}}(E_{(3)} + E_{(4)})}{\sqrt{2}}
$$

$$
\psi_1{}^3 = \frac{(B_{(1)} - E_{(2)}) \cosh\left[\frac{F(t)}{2}\right]}{\sqrt{2}}; \; \psi_1{}^4 = \frac{(B_{(2)} + E_{(1)}) \cosh\left[\frac{F(t)}{2}\right]}{\sqrt{2}}
$$

$$
\psi_1{}^5 = -\frac{(B_{(3)} + B_{(4)}) \cosh\left[\frac{F(t)}{2}\right]}{\sqrt{2}}; \; \psi_1{}^6 = \frac{e^{\frac{F(t)}{2}}(-E_{(3)} + E_{(4)})}{\sqrt{2}}
$$

$$
\psi_1{}^7 = \frac{(B_{(1)} + E_{(2)}) \cosh\left[\frac{F(t)}{2}\right]}{\sqrt{2}}; \; \psi_1{}^8 = \frac{(B_{(2)} - E_{(1)}) \cosh\left[\frac{F(t)}{2}\right]}{\sqrt{2}},
$$

(3)

when $\mathbf{u}$ is given by Eq.[1]. (The field equations guarantee that $E_{(4)} = 0 = B_{(4)}$.)

3 Non-equilibrium birth

In this model we **assume** that at the onset of creation at $t = 0$ the particles/fields created by the first Big Bang are **not** in thermal equilibrium. We resolve the charge density of "all" electrically charged matter as $\rho_{matter}^Q = \rho_{matter}^Q(t) = |e| \int_{-\infty}^{\infty} \left[(-1)\rho_{electron}(t; \overrightarrow{v}) + (-\frac{1}{3})\rho_{down}(t; \overrightarrow{v}) + (+\frac{2}{3})\rho_{up}(t; \overrightarrow{v}) \right] d^3 \overrightarrow{v}$. Here $\rho_{\mathrm{species}}(t; \overrightarrow{v}) = \int_{-\infty}^{\infty} \rho_{\mathrm{species}}(t; \overrightarrow{v}; \overrightarrow{x}) d^3 \overrightarrow{x}$ denotes an associated energy density distribution function and the integral is over the hypersurface $t = 0$. Let $\rho_{anti-matter}^Q = \rho_{anti-matter}^Q(t)$ denote the charge density of "all"

electrically charged anti-leptons and anti-quarks (in this paper this means only positrons, and anti-up and anti-down quarks). We assume that $\rho^Q_{matter} + \rho^Q_{anti-matter} = 0$, so that the newly created universe has zero net charge (in fact, one of the Euler-Lagrange equations demands that this net charge be zero). However we assume that while the spatial components of the sum of the linear momenta of all particles and fields sum to zero, at comoving time t = 0 a non-vanishing average current drift velocity $\overrightarrow{J}_D$ instantaneously exists that satisfies

$$\overrightarrow{0} \neq \overrightarrow{J}_D =$$
$$\int_{-\infty}^{\infty} \overrightarrow{v}\, |e| \left\{ \left[(-1)\rho_{electron}(t; \overrightarrow{v}) + (-\frac{1}{3})\rho_{down}(t; \overrightarrow{v}) + (+\frac{2}{3})\rho_{up}(t; \overrightarrow{v}) \right] \right.$$
$$\left. - \left[(-1)\rho_{positron}(t; \overrightarrow{v}) + (-\frac{1}{3})\rho_{anti-down}(t; \overrightarrow{v}) + (+\frac{2}{3})\rho_{anti-up}(t; \overrightarrow{v}) \right] \right\}\, d^3\overrightarrow{v}.$$
$$(4)$$

It is assumed that the comoving $z = x^3$-axis is aligned with the average drift velocity at $t = 0$. (When the field equations are solved it is found that this model predicts that $\overrightarrow{J}_D(t) \to \overrightarrow{0}$ very shortly after $t = 0$.) $\overrightarrow{J}_D \neq \overrightarrow{0}$ sources a non-isotropic contribution to the total energy momentum tensor near $t = 0$. In order to make the total energy momentum tensor homogeneous and isotropic this non-isotropic term must be canceled by an additional non-isotropic term (see Eq.[12]) that must be incorporated into the Lagrangian.

In passing we note that in the 1930s, Robertson and Walker independently proved that there are only three possible spacetime metrics for a universe that is homogeneous and isotropic. A spacetime having one of these three spacetime metrics must be associated with an energy-momentum tensor that is homogeneous and isotropic. Coley and Tupper [6] have shown that a fluid with an axial velocity may possess a homogeneous and isotropic energy-momentum tensor, and therefore source a Robertson and Walker spacetime geometry. In this paper the total energy-momentum tensor is homogeneous and isotropic, although its constituents are generally not.

4 Lagrangian formulation of the problem

Let $\mathbb{E} \equiv det\left(E_\alpha^{(\mu)}\right) = \sqrt{-g} \neq 0$, R denote the Ricci scalar and $\mathcal{L}_{\text{Einstein-Hilbert}} = \mathbb{E}R$. We write the total Lagrangian density for this problem as

$$\mathcal{L} = \frac{1}{2}\frac{1}{8\pi G}\left(\mathcal{L}_{\text{Einstein-Hilbert}} + \mathcal{L}_{\mathbf{u}}\right)$$
$$+ \mathcal{L}_{EM} + \mathcal{L}_{Fermions} + \mathcal{L}_{\varrho p}.$$
$$(5)$$

Here $\mathcal{L}_{\mathbf{u}}$ is the Lagrangian for the unit spinor field u^a, $\mathcal{L}_{EM}$ is the Lagrangian for the electromagnetic field Ψ_2, $\mathcal{L}_{Fermions}$ is comprised of six Lagrangians that govern the dynamics of the massive and electrically charged electron,

up quark and down quark, and their antiparticle fields. These fields are not second-quantized. $\mathcal{L}_{\varrho p}$ is somewhat similar to the familiar Lagrangian for an effective fluid that is characterized by an effective proper energy density $\varrho(t)$ and an effective pressure $\mathrm{p}(t)$. In this model both the energy density $\varrho(t)$ and the pressure $\mathrm{p}(t)$ are determined dynamically by the field equations.

4.1 Lagrangian for the unit field u^a

Let $\gamma^\alpha = E^\alpha_{(\mu)}\,\tau^{(\mu)}\,\tau^{(7)}$. The Lagrangian for the unit field u^a is

$$\mathcal{L}_{\mathbf{u}} = \mathbb{E}\,\frac{1}{2\sqrt{4\,\pi\,G\,\widetilde{\mathbf{u}}\,\sigma\,\mathbf{u}}}\,\widetilde{\mathbf{u}}\,\sigma\,\tau^{(5)}\tau^{(6)}\gamma^\alpha\,\mathbf{u}_{|\alpha}. \tag{6}$$

The the σ and τ matrices are defined in the Appendix; a representation of the 8×8 σ matrix is given in Eq.[50] and a representation of the τ matrices is given in Eq.[89]. The vertical bar denotes spinor covariant derivative (see Appendix 9). We note that the terms $\tau^{(5)}\tau^{(6)}$ (and $\tau^{(7)}$, below) break the $\overline{SO(4,4;\mathbb{R})}$ symmetry down to $\overline{SO(3,1;\mathbb{R})}$.

4.2 Lagrangian for the electromagnetic field Ψ_2

The interaction with the electromagnetic field possesses local $U(1)$ invariance and incorporates the covariant gradient operator

$$D_\alpha = \mathbb{I}_{8\times 8}\,\nabla_\alpha\,-\,\tau^C\left(\frac{\mathbf{u}}{\sqrt{\widetilde{\mathbf{u}}\,\sigma\,\mathbf{u}}}\right)_{|\,\alpha}\,\frac{\widetilde{\mathbf{u}}}{\sqrt{\widetilde{\mathbf{u}}\,\sigma\,\mathbf{u}}}\,\sigma\,\overline{\tau}_C. \tag{7}$$

The local $U(1)$-invariant electromagnetic Lagrangian $\mathcal{L}_{EM}$ for this problem is [21]

$$\begin{aligned}
\frac{1}{\mathbb{E}}\,\mathcal{L}_{EM} &= \frac{1}{2}\widetilde{\Psi_1}\,\sigma\,\Psi_1\,+\,\widetilde{\Psi_1}\,\sigma\,\mathbb{P}_\perp\,\overline{\tau}^\alpha\,D_\alpha\left(\Psi_2\right) \\
&= \frac{1}{2}\,\widetilde{\Psi_1}\,\sigma\,\Psi_1\,+\,\widetilde{\Psi_1}\,\sigma\,\mathbb{P}_\perp\,\overline{\tau}^\alpha\,\Psi_{2|\,\alpha} \\
&\quad -\,\widetilde{\Psi_1}\,\sigma\,\mathbb{P}_\perp\,\overline{\tau}^\alpha\left(\tau^C\left(\frac{\mathbf{u}}{\sqrt{\widetilde{\mathbf{u}}\,\sigma\,\mathbf{u}}}\right)_{|\,\alpha}\,\frac{\widetilde{\mathbf{u}}}{\sqrt{\widetilde{\mathbf{u}}\,\sigma\,\mathbf{u}}}\,\sigma\,\overline{\tau}_C\right)\Psi_2
\end{aligned} \tag{8}$$

This Lagrangian gives the conventional interacting local $U(1)-$invariant Maxwell equations in a flat spacetime [21].

4.3 Fermion $\mathbf{\Phi}$ Lagrangian

For $\mathbf{\Phi}$ = {fields associated to the electron, up quark and down quark, and their antiparticles}, there are terms in the total Lagrangian of the form

$$
\begin{aligned}
\mathcal{L}_{Fermion}\left[\mathbf{\Phi}\right] &= \frac{1}{2}\mathbb{E}\,\mathbf{\Phi}\,\sigma\,\left[\gamma^{\mu}\,\mathbf{\Phi}_{|\mu}\,+\,\mathbb{M}\tau^{(5)}\tau^{(6)}\tau^{(7)}\mathbf{\Phi}\right] \\
&\quad + \frac{1}{2}\mathbb{E}\,\mathbf{\Phi}\,\sigma\,\tau^{(\mu)}\,\left(\xi\frac{1}{\sqrt{\widetilde{\mathbf{u}}\,\sigma\,\mathbf{u}}}\widetilde{\mathbf{u}}\,\sigma\,\bar{\tau}_{(\mu)}\,\Psi_2\right)\tau^{(5)}\tau^{(6)}\tau^{(7)}\,\mathbf{\Phi} \\
&\equiv \mathcal{L}_F\left[\mathbf{\Phi}\right].
\end{aligned}
\tag{9}
$$

Here $\xi = \xi(\mathbf{\Phi})$ characterizes the electric charge carried by the fields, and takes values $\frac{\xi}{|e|} = -1, \frac{2}{3}, -\frac{1}{3}, 1, -\frac{2}{3}$ and $\frac{1}{3}$ for $\mathbf{\Phi} = (e, u, d, \overline{e}, \overline{u}, \overline{d})$. The charge operator is not diagonal and is given by

$$
Q = \xi\,\tau^{(5)}\,\tau^{(6)}\,\tau^{(7)}\,.
\tag{10}
$$

The mass parameter $\mathbb{M}$ takes values $\mathbb{M} = \mathbb{M}_e, \mathbb{M}_u, \mathbb{M}_d, \mathbb{M}_{\overline{e}} = \mathbb{M}_e, \mathbb{M}_{\overline{u}} = \mathbb{M}_u$ and $\mathbb{M}_{\overline{d}} = \mathbb{M}_d$. The Fermion Lagrangian is

$$
\begin{aligned}
\mathcal{L}_{Fermion} &= \Lambda_m\left(\mathcal{L}_F\left[\mathbf{\Phi_e}\right] + \mathcal{L}_F\left[\mathbf{\Phi_u}\right] + \mathcal{L}_F\left[\mathbf{\Phi_d}\right]\right) \\
&\quad + \Lambda_a\left(\mathcal{L}_F\left[\overline{\mathbf{\Phi_e}}\right] + \mathcal{L}_F\left[\overline{\mathbf{\Phi_u}}\right] + \mathcal{L}_F\left[\overline{\mathbf{\Phi_d}}\right]\right).
\end{aligned}
\tag{11}
$$

Here $\Lambda_m = \Lambda_{matter}$ and $\Lambda_a = \Lambda_{anti-matter}$ are order parameters that keep track of matter/anti-matter contributions to the field equations.

In a flat Minkowski spacetime the Euler-Lagrange equations associated to this Lagrangian imply the same mass/four-momentum dispersion relationship, and minimal electromagnetic coupling, as does the conventional Dirac equation.

4.4 Density and pressure terms

Our approach to implementing the consequences of the assumption of the non-equilibrium birth of the universe discussed in Section [3], with its preferred (but short lived) initial direction, is to employ

$$
\mathcal{L}_{\varrho p} = \mathbb{E}\,T^{\alpha}_{(\mu)}\,E^{(\mu)}_{\alpha},
\tag{12}
$$

where

$$
T^{\alpha}_{(\mu)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & p(t) & 0 \\ 0 & 0 & 0 & \varrho(t) \end{pmatrix}
\tag{13}
$$

in the Lagrangian. Here $\varrho(t)$ and $p(t)$ denote the energy density and pressure parameters, but they are not identified with a fluid. There is no *a priori* equation of state relating $p(t)$ and $\varrho(t)$ because $p(t)$ is determined dynamically from the field equations. $\dot{\varrho}(t)$ may be computed as follows. We solve the field equations for $\dot{a}(t)$ [$= DA(a(t), F(t), \varrho(t), \ldots)$, say] and $\ddot{a}(t)$ [$= DDA(a(t), F(t), \varrho(t), \ldots)$, say] and require that $\frac{d}{dt} DA(a(t), F(t), \varrho(t), \ldots) = DDA(a(t), F(t), \varrho(t), \ldots)$ when the field equations are satisfied. $\dot{\varrho}(t)$ appears linearly in the last equation, and can always be solved for.

It should be emphasized that the total energy-momentum tensor for this problem is homogeneous and isotropic, although the constituents such as $T_{(\mu)}^{\alpha}$ of Eq.[13] are generally not.

A possible alternative approach to employing Eq.[12] in the total Lagrangian $\mathcal{L}$ is to include a self-interacting scalar field Υ in $\mathcal{L}$ using

$$\mathcal{L}_{\Upsilon} = -\mathbb{E} \left\{ \frac{1}{2} g^{\alpha\beta} \partial_{\alpha}\Upsilon \partial_{\beta}\Upsilon + V(\Upsilon) \right\}. \tag{14}$$

Let S_{Υ} denote the action of the scalar field. The energy-momentum tensor for the scalar field is

$$T_{\mu\nu}^{(\Upsilon)} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{\Upsilon}}{\delta g^{\mu\nu}} = \partial_{\mu}\Upsilon \partial_{\nu}\Upsilon - g_{\mu\nu} \left\{ \frac{1}{2} g^{\alpha\beta} \partial_{\alpha}\Upsilon \partial_{\beta}\Upsilon + V(\Upsilon) \right\}. \tag{15}$$

Assuming the FLRW metric for $g_{\alpha\beta}$ and restricting to the case of a homogeneous field $\Upsilon(t)$, this energy-momentum tensor $T_{\mu\nu}^{(\Upsilon)}$ takes the form of a perfect fluid, with the energy density and pressure given by

$$\varrho = \frac{1}{2}\dot{\Upsilon}^2 + V(\Upsilon)$$
$$p = \frac{1}{2}\dot{\Upsilon}^2 - V(\Upsilon). \tag{16}$$

Therefore, unfortunately, this method cannot lead to a cancelation of a non-isotropic term in the total energy-momentum tensor, because $T_{\mu\nu}^{(\Upsilon)}$ is homogeneous and isotropic.

5 Field Equations

All of the variations of the Lagrangian with respect to the fields are first performed for the most general forms of the field functions, and then the special functional forms $u^a = \frac{1}{\sqrt{2}} \left(0, e^{-\frac{F(t)}{2}}, 0, 0, 0, e^{\frac{F(t)}{2}}, 0, 0 \right)$, $E_{\alpha}^{(\mu)} = \text{diag}(a(t), a(t), a(t), 1)$, and so on are substituted into the resulting Euler-Lagrange equations. Unfortunately the full set of field equations is too lengthy to print (please see below for an exception). However one combination of variational derivatives is simple

and important:

$$\frac{\delta\mathcal{L}}{\delta u^2(t)} + e^{F(t)}\frac{\delta\mathcal{L}}{\delta u^6(t)}$$

$$= \frac{e^{\frac{F(t)}{2}}\,a(t)^2\,\left(3\,\cosh\left[F(t)\right]\dot{a}(t) + a(t)\sinh[F(t)]\,\dot{F}(t)\right)}{8\sqrt{2}}$$

$$= \frac{1}{8\sqrt{2}}\,e^{\frac{F(t)}{2}}\,\frac{d}{dt}\left\{a(t)^3\cosh\left[F(t)\right]\right\}$$

$$= 0 \tag{17}$$

Summarizing results, this model predicts that

$$a(t)^3\cosh\left[F(t)\right] = a(0)^3\cosh\left[F(0)\right] = \text{constant}. \tag{18}$$

$F(t) = 0$ is one characterization of the Minkowski spacetime of Einstein's special relativity. Since $F(t) \to 0$ characterizes the end of inflation we conclude that $\frac{a(t_{\text{end of inflation}})}{a(0)} = \cosh\left[F(0)\right]^{1/3}$. $F(0) \approx \pm 3 \times N$ implies that inflation lasts for about N e-folds.

The Euler-Lagrange equations imply that the magnetic field $\mathbf{B}$ vanishes but the electric field $\mathbf{E}$ has non-vanishing z-component (that exponentially vanishes).

5.1 Special case

For the special case without Standard Model degrees of freedom, obtained by setting to zero the Fermion degrees of freedom and the electromagnetic Ψ_2 and Ψ_1 spinors, the Einstein sector of the field equations is

$$\frac{\delta\mathcal{L}}{\delta E_\alpha^{(\mu)}} = \text{diag}\begin{pmatrix} a(t)\left[a(t)\{a(t)p(t) + \varrho(t)\} - a''(t)\right] - \frac{1}{2}a'(t)^2 \\ a(t)\left[a(t)\{a(t)p(t) + \varrho(t)\} - a''(t)\right] - \frac{1}{2}a'(t)^2 \\ a(t)\left[a(t)\{2a(t)p(t) + \varrho(t)\} - a''(t)\right] - \frac{1}{2}a'(t)^2 \\ p(t)a(t)^4 + 2\varrho(t)a(t)^3 - \frac{3}{2}a'(t)^2 a(t) \end{pmatrix}. \tag{19}$$

With

$$g(t) \equiv \text{sech}[F(t)], \tag{20}$$

the field equations imply that

$$p(t) = 0 \tag{21}$$

$$\dot{a}(t) = \pm\frac{2}{3}a(t)\sqrt{3\,\varrho(t)} \tag{22}$$

$$\ddot{a}(t) = \frac{1}{3}a(t)\varrho(t) \tag{23}$$

$$\dot{F}(t) = \mp 2\sqrt{3\,\varrho(t)}\,\coth(F(t)) \tag{24}$$

$$\Leftrightarrow \dot{g}(t) = \pm 2\sqrt{3\,\varrho(t)}\,g(t), \tag{25}$$

where Eq.[17] and Eq.[22] have been used to arrive at Eq.[24] . Differentiating Eq.[22] with respect to t and setting the result equal to Eq.[23] yields

$$\dot{\varrho}(t) = \mp\sqrt{3}\,\varrho(t)^{3/2}. \tag{26}$$

The field equations Eqs.[22],[25] and [26] imply that

$$\ddot{g}(t) = 9\,\varrho(t)\,g(t) \tag{27}$$

$$\ddot{\varrho}(t) = \frac{9}{2}\,\varrho(t)^2. \tag{28}$$

5.1.1 Integrals of the motion

Dividing Eqs.[22] and [26] by $\dot{F}(t)$ yields

$$\frac{\partial}{\partial F}a(F) = -\frac{1}{3}\,a(F)\,\tanh(F) \tag{29}$$

$$\frac{\partial}{\partial F}\varrho(F) = \frac{1}{2}\,\varrho(F)\,\tanh(F) \tag{30}$$

Integrating, two independent integrals of the motion obtained:

$$a^3\,\cosh\left(F\right) = \text{constant} = a_i^3\,\cosh\left(F_i\right) \tag{31}$$

$$\varrho^2\,\mathrm{sech}(F) = \text{constant} = \varrho_i{}^2\,\mathrm{sech}(F_i), \tag{32}$$

where F_i, a_i and ϱ_i denote initial values. These relations imply that $\varrho^2\,a^3 = $ constant.

5.1.2 Integration of the field equations Eqs.[22],[25] and [26]

Eq.[24] and Eq.[32] imply that $\dot{g}(t) = \pm 2g(t)^{3/4}\sqrt{3\,\varrho_i}\,\sqrt[4]{\mathrm{sech}\left(F_i\right)}$. The differential equation for $F(t)$ has a regular singular point at $F(t) = 0$ and is not analytic at $F(t) = \infty$. Consequently $\dot{g}(t)$ is discontinuous at times when $g(t) = 1$, and switches Riemann surfaces when $g(t)$ crosses zero.

Shifting the comoving time origin from zero, the equations for a, ϱ and $g(t)$ may be easily integrated and the solutions expressed as

$$a = a_i\,\sqrt[3]{\mathrm{sech}(F)\cosh(F_i)} = a_i\,\sqrt[3]{\frac{g}{g_i}} \tag{33}$$

$$\varrho = \varrho_i\,\sqrt{\cosh(F)\mathrm{sech}(F_i)} = \varrho_i\,\sqrt{\frac{g_i}{g}} \tag{34}$$

$$
\begin{aligned}
g(t) = {}&\\
\mathrm{sech}(F(t)) = {}&\\
&\begin{cases}
\mathrm{sech}\left(F_i\right)\left(t\,\frac{\sqrt{3\,\varrho_i}}{2} - \sqrt[4]{\cosh\left(F_i\right)}\right)^4, & 0 \le t\,\frac{\sqrt{3\,\varrho_i}}{2} < \sqrt[4]{\cosh\left(F_i\right)} \\[2mm]
\mathrm{sech}\left(F_i\right)\left(t\,\frac{\sqrt{3\,\varrho_i}}{2} + \sqrt[4]{\cosh\left(F_i\right)}\right)^4, & -\sqrt[4]{\cosh\left(F_i\right)} < t\,\frac{\sqrt{3\,\varrho_i}}{2} \le 0
\end{cases}\\[3mm]
= {}&\, g\left(t + 4\,\sqrt[4]{\frac{\cosh\left(F_i\right)}{9\,\varrho_i{}^2}}\right),
\end{aligned}
\tag{35}
$$

where $F(0) = 0 \Leftrightarrow g(0) = 1$ and F_i ($\neq F(0)$ in general), g_i, a_i and ϱ_i denote initial values. Therefore, without Standard Model degrees of freedom, $F(t)$ is periodic in t with period

$$T = 4 \sqrt[4]{\frac{\cosh\left(F_i\right)}{9\,\varrho_i{}^2}} = \frac{4}{\sqrt[4]{9\,\varrho_i{}^2\,g_i}}, \tag{36}$$

and angular frequency

$$\omega = \frac{\pi}{2} \sqrt[4]{\frac{9\,\varrho_i{}^2}{\cosh\left(F_i\right)}} = \frac{\pi}{2} \sqrt[4]{9\,\varrho_i{}^2\,g_i}, \tag{37}$$

that depend on initial conditions. The integral of the motion $\varrho^2 \operatorname{sech}(F) = \varrho^2 g = \text{constant} = \frac{1}{9}\left(\frac{2\omega}{\pi}\right)^4$. Note that the scale factor will make the transition from a_i to $a_{max} = a(0) = a_i \sqrt[3]{\cosh(F_i)} = a_i \frac{1}{\sqrt[3]{g_i}}$ in a finite time that is less than $\frac{1}{2}T$. a_{max} can be made as large as one likes by reducing g_i, and T can be made as small as one chooses by increasing $\varrho_i{}^2 g_i$. To address the horizon problem in a model that includes material, all that is needed is for a to be small enough for enough time for the system to come to thermal equilibrium. The fact that a vanishes guarantees this. Even if we can find a way to keep a nonzero, the fact that a oscillates over many periods will guarantee that ultimately a system is brought to thermal equilibrium.

Alternatively, to solve the field equations we may define

$$\begin{aligned}
\tau(t) &= \frac{t}{4} \sqrt[4]{9\varrho(t)^2 g(t)} \\
a(t) &= A[\tau(t)]G[\tau(t)]^{4/3} \\
g(t) &= G[\tau(t)]^4 \\
\varrho(t) &= R[\tau(t)]G[\tau(t)]^{-2}.
\end{aligned} \tag{38}$$

The field equations Eqs.[22],[25] and [26], re-expressed as functions of τ on a single Riemann surface, are locally

$$\frac{d}{d\tau}G(\tau) = \pm 2, \frac{d}{d\tau}A(\tau) = 0 = \frac{d}{d\tau}R(\tau). \tag{39}$$

This special case is unphysical because of Eq.[4], $\vec{J}_D(0) \neq \vec{0}$. Nevertheless this case demonstrates that the physical $F(t)$ may be almost periodic in t.

5.1.3 Lagrangian for the special case

A Lagrangian $\mathbb{L}_0$ may be defined for this special case. Let

$$\begin{aligned}
\mathbb{L}_0 = \frac{1}{4} \Bigg\{ &\left[\left(\dot{a}(t) - \frac{2}{3}\,a(t)\,\sqrt{3\,\varrho(t)}\right)\left(\dot{a}(t) + \frac{2}{3}\,a(t)\,\sqrt{3\,\varrho(t)}\right)\right]^2 \\
&+ \left[\left(\dot{g}(t) - 2g(t)\sqrt{3\,\varrho(t)}\right)\left(\dot{g}(t) + 2g(t)\sqrt{3\,\varrho(t)}\right)\right]^2
\end{aligned}$$

$$+ \left[\left(\dot{\varrho}(t) + \sqrt{3\,\varrho(t)}\,\varrho(t) \right) \left(\dot{\varrho}(t) - \sqrt{3\,\varrho(t)}\,\varrho(t) \right) \right]^2 \Bigg\}.$$

$$(40)$$

Clearly $\mathbb{L}_0 = 0$ when the field equations Eqs.[22],[25] and [26] are satisfied, which is the unique absolute minimum for this functional. The Euler-Lagrange equations associated to $\mathbb{L}_0$ are satisfied when the field equations Eqs.[22],[25] and [26] (and their higher derivatives, Eqs.[27] and [28]) are satisfied. Put $x^j = (a, g, \varrho)$, $j = 1, 2, 3$. The canonical momenta are $p_j = \frac{\partial}{\partial \dot{x}^j} \mathbb{L}_0$. The Jacobian $\frac{\partial^2}{\partial \dot{x}^k \partial \dot{x}^j} \mathbb{L}_0$ is non-zero, so we can solve for the $\dot{x}^j$ as functions $\dot{x}^j = v^j(t, x^j, p_j)$ of the canonical variables (t, x^j, p_j).

The canonical Hamiltonian function is given by $\mathbb{H}(x^j, p_j) = -\mathbb{L}_0 + p_j \dot{x}^j$, and in this case is independent of t. Here it is assumed that the $\dot{x}^j$, on the right-hand-side of this definition are the inverse functions $\dot{x}^j = v^j(t, x^j, p_j)$. $\mathbb{L}_0$ is of class C^∞ if we exclude the hyperplane $\varrho = 0$ in $(a, g, \varrho, p_a, p_g, p_\varrho) -$ phase space. It follows that the p_j are continuously differentiable functions of $(t, x^j, \dot{x}^j)$ in the same region. However the inverse functions $v^j(t, x^j, p_j)$ and $\mathbb{H}(t, x^j, p_j)$ are not continuous everywhere in phase space. These functions have branch points and cuts in the six-dimensional phase space, in virtue of the roots of the combined powers of the momenta and coordinates that arise in solving for the $\dot{x}^j$. For example, when $\dot{a} > 0$ and $0 < p_a << 1$ then

$\dot{a} = \frac{(8+8i\sqrt{3})a^2\varrho - 2(-1)^{2/3}2^{1/3}\left(-27p_a + \sqrt{729p_a{}^2 - 256a^6\varrho^3}\right)^{2/3}}{6\ 2^{2/3}\left(-27p_a + \sqrt{729p_a{}^2 - 256a^6\varrho^3}\right)^{1/3}} \approx \frac{3p_a}{8a^2\varrho} + \frac{2}{3}a\sqrt{3\varrho}$; also $\dot{g} \approx \frac{p_g}{24g^2\varrho} + 2\,g\sqrt{3\varrho}$, $\dot{\varrho} \approx \frac{p_\varrho}{6\varrho^3} - \sqrt{3\varrho}\varrho$, which are real in the appropriate region of phase space. These relations are reminiscent of a charged particle moving in an external vector potential proportional to $\sqrt{3\varrho}(\frac{2}{3}a, 2\,g, -\varrho)$ with respect to some curvilinear coordinate system. Near the origin of momentum space

$$\mathbb{H}_0 \approx \frac{3p_a{}^2}{16a^2\varrho} + \frac{p_\varrho{}^2}{12\varrho^3} + \frac{p_g{}^2}{48g^2\varrho} + \sqrt{3\varrho}\left(\frac{2}{3}ap_a + 2gp_g - p_\varrho\varrho\right). \qquad (41)$$

Once the Hamiltonian $\mathbb{H}_0$ has been defined and operator ordering problems have been resolved one may in principle quantize the system and study the eigenstates of $\mathbb{H}_0$. An understanding of the quantized system may aid in the interpretation of the periodic solutions of the field equations.

5.2 Full problem

For the full problem we find that $\dot{F}(t)$ has a regular singular points at $F(t) = 0$ and is not analytic as $F(t) \to \infty$. After one fixes the initial values at $t = 0$ of the Fermions and the electromagnetic Ψ_2 and Ψ_1 fields, with $\varrho(0) > 0$, then our numerical experiments verify that $\varrho(t) \geq 0$ and that $\frac{\varrho(t)^2}{\cosh[F(t)]} \approx \frac{\varrho(0)^2}{\cosh[F(0)]}$. The rate of change of $\varrho(t)$ when $\dot{a} > 0$ compared to this rate when $\dot{a} < 0$ depends on Λ_{matter} and $\Lambda_{anti-matter}$, as follows (using Eq.[10]):

$$\dot{\varrho}(t)|_{\dot{a}\,>\,0} = -\dot{\varrho}(t)|_{\dot{a}\,<\,0} \;+\; \sqrt{2}\,\psi_2^1(t)\,e^{-\frac{F(t)}{2}}\;\{$$

$$\Lambda_m\left(-\frac{1}{3}\,\mathbb{M}_d\left[\widetilde{\Phi_d(t)}\,\tau^4\,Q\,\Phi_d(t)\right] - \mathbb{M}_e\left[\widetilde{\Phi_e(t)}\,\tau^4\,Q\,\Phi_e(t)\right] + \frac{2}{3}\,\mathbb{M}_u\left[\widetilde{\Phi_u(t)}\,\tau^4\,Q\,\Phi_u(t)\right]\right)$$

$$+\;\Lambda_a\left(\frac{1}{3}\,\mathbb{M}_d\left[\widetilde{\overline{\Phi_d}(t)}\,\tau^4\,Q\,\overline{\Phi_d}(t)\right] + \mathbb{M}_e\left[\widetilde{\overline{\Phi_e}(t)}\,\tau^4\,Q\,\overline{\Phi_e}(t)\right] - \frac{2}{3}\,\mathbb{M}_u\left[\widetilde{\overline{\Phi_u}(t)}\,\tau^4\,Q\,\overline{\Phi_u}(t)\right]\right)\}$$

$$-\;\frac{3}{\sqrt{2}}\,\xi\,\psi_1^2(t)\,e^{\frac{F(t)}{2}}\;\{$$

$$\Lambda_m\left(-\frac{1}{3}\left[\widetilde{\Phi_d(t)}\,\tau^4\,\tau^3\,\Phi_d(t)\right] - \left[\widetilde{\Phi_e(t)}\,\tau^4\,\tau^3\,\Phi_e(t)\right] + \frac{2}{3}\left[\widetilde{\Phi_u(t)}\,\tau^4\,\tau^3\,\Phi_u(t)\right]\right)$$

$$+\;\Lambda_a\left(\frac{1}{3}\left[\widetilde{\overline{\Phi_d}(t)}\,\tau^4\,\tau^3\,\overline{\Phi_d}(t)\right] + \left[\widetilde{\overline{\Phi_e}(t)}\,\tau^4\,\tau^3\,\overline{\Phi_e}(t)\right] - \frac{2}{3}\left[\widetilde{\overline{\Phi_u}(t)}\,\tau^4\,\tau^3\,\overline{\Phi_u}(t)\right]\right)\} \tag{42}$$

The physics governing the escape of F and a from the periodic trap has not been discovered. If the value of the energy density $\varrho(t)$ decreases to the point where

$$\lim_{F(t)\to 0}\frac{\varrho(t)}{F(t)^2} = 0$$

then $\lim_{F(t)\to 0}\dot{F}(t) = 0$, as well as $\lim_{F(t)\to 0}\frac{d^n F(t)}{d\,t^n} = 0,\; n = 2, 3, ...,$ and therefore $F \to 0$. a is controlled by F [because $a^3\cosh(F) = $ constant] and will escape the periodic trap and very rapidly transition to a constant $a(t_{\text{end of inflation}}) \approx a_1,$ since $F(t) \equiv 0\;\forall\, t > t_{\text{end of inflation}}$. However the governing physics may have more similarity with quantum mechanical tunneling through a barrier than the decay of the controlling parameter $\varrho(t)$ to zero. The investigation is ongoing.

In numerical experiments we seek a solution of the form $u^a = \frac{1}{\sqrt{2}}\left\{\delta_2^a\,e^{-\frac{F(t)}{2}} + \delta_6^a\,e^{\frac{F(t)}{2}}\right\}$, (with $F(t) \equiv 0$ characterizing the Minkowski spacetime of Einstein's special relativity, but not uniquely), $\Psi_2^a(t) = \left\{\delta_1^a + \delta_5^a\,e^{-F(t)}\right\}\psi_2^1(t)$ and $\Psi_1^a(t) = \left\{\delta_2^a - \delta_6^a\,e^{F(t)}\right\}\psi_1^2(t)$. The fermions are assumed to have vanishing components Φ^a for $a = 3, 4, 7, 8$. We find that

$$\dot{g}(t) = \pm g(t)\,\sqrt{6}\left[2\varrho(t) - \psi_1^2(t)^2 + \frac{1}{3}\sqrt{2}\xi\psi_2^1(t)\,(\right.$$

$$\Lambda_a\left(-\left(\bar{\Phi}_d^1(t)\right)^2 - \left(\bar{\Phi}_d^2(t)\right)^2 + \left(\bar{\Phi}_d^5(t)\right)^2 + \left(\bar{\Phi}_d^6(t)\right)^2 - 3\left(\bar{\Phi}_e^1(t)\right)^2 - 3\left(\bar{\Phi}_e^2(t)\right)^2\right.$$

$$+\;3\left(\bar{\Phi}_e^5(t)\right)^2 + 3\left(\bar{\Phi}_e^6(t)\right)^2 + 2\left(\left(\bar{\Phi}_u^1(t)\right)^2 + \left(\bar{\Phi}_u^2(t)\right)^2 - \left(\bar{\Phi}_u^5(t)\right)^2 - \left(\bar{\Phi}_u^6(t)\right)^2\right))$$

$$+\;\Lambda_m\left(\left(\Phi_d^1(t)\right)^2 + \left(\Phi_d^2(t)\right)^2 - \left(\Phi_d^5(t)\right)^2 - \left(\Phi_d^6(t)\right)^2 + 3\left(\Phi_e^1(t)\right)^2 + 3\left(\Phi_e^2(t)\right)^2\right.$$

$$-\;3\left(\Phi_e^5(t)\right)^2 - 3\left(\Phi_e^6(t)\right)^2 - 2\left(\Phi_u^1(t)\right)^2 - 2\left(\Phi_u^2(t)\right)^2 + 2\left(\Phi_u^5(t)\right)^2 + 2\left(\Phi_u^6(t)\right)^2))$$

$$+\;\Lambda_m\left(\mathbb{M}_d\left(\Phi_d^2(t)\Phi_d^5(t) - \Phi_d^1(t)\Phi_d^6(t)\right)\right.$$

$$+\;\mathbb{M}_e\left(\Phi_e^2(t)\Phi_e^5(t) - \Phi_e^1(t)\Phi_e^6(t)\right) + \mathbb{M}_u\left(\Phi_u^2(t)\Phi_u^5(t) - \Phi_u^1(t)\Phi_u^6(t)\right))$$

$$+\;\Lambda_a\left(\mathbb{M}_d\left(\bar{\Phi}_d^2(t)\bar{\Phi}_d^5(t) - \bar{\Phi}_d^1(t)\bar{\Phi}_d^6(t)\right)\right.$$

$$+ \; \mathbb{M}_e \left(\bar{\Phi}_e^2(t) \bar{\Phi}_e^5(t) - \bar{\Phi}_e^1(t) \bar{\Phi}_e^6(t) \right) + \mathbb{M}_u \left(\bar{\Phi}_u^2(t) \bar{\Phi}_u^5(t) - \bar{\Phi}_u^1(t) \bar{\Phi}_u^6(t) \right) \big)]^{1/2} ,$$

(43)

where $\pm$ according to $\dot{a} \gtrless 0$.

Initial values are $a(0) = 10^{-24}$, $F(0) = -180, \varrho(0) = 10^{40}, \psi_2^1(0) = 0 = \psi_1^2(0)$, $\Phi_e(0) = \frac{1}{\sqrt{2}}(1,0,0,0,1,0,0,0) = \Phi_d(0)$ (and their antiparticles) and $\Phi_u(0) = \sqrt{2}\frac{1}{\sqrt{2}}(1,0,0,0,1,0,0,0) = \overline{\Phi}_u(0)$. Figures [4-8] summarize the results of representative numerical integrations of the coupled field equations. We note that experiments with $\psi_2^1(0) \sim 10^{40} \sim \left\{ \psi_1^2(0) \right\}^2$ have been performed using arbitrary precision floating point numbers to avoid over/under-flows near $t = 0$. Calculations employing 450 bits for each floating point number seem to maintain accuracy. However, to date, high precision calculations (running several days) have only evolved the system to $t = 10^{-98} T$, where T is the expected period and is of order 1.

6 Conclusion

We have studied a new cosmological model that is based on a Lagrangian and couples Einstein, electromagnetic (with quark and lepton sources) and unit **u** field [21] sectors. We have focussed on a particular solution of coupled Euler-Lagrange field equations that employs a flat ($k = 0$) universe Friedmann-Lemaître-Robertson-Walker (FLRW) line element with scale factor $a = a(t)$ [comoving coordinates] for the Einstein sector. The solution of the field equations yields a scale factor $a(t)$ that oscillates almost periodically, and admits the first integral $a(t)^3 \cosh\left[F(t)\right] = a(0)^3 \cosh\left[F(0)\right] = $ constant. Since $F(t) \equiv 0 \, \forall t > t_{\text{end of inflation}}$ characterizes the end of inflation, $\frac{a(t_{\text{end of inflation}})}{a(0)} = \cosh\left[F(0)\right]^{1/3}$, where $\cosh\left[F(t)\right] = \widetilde{\mathbf{u}}\mathbf{u} = \left[u^2(t)\right]^2 + \left[u^6(t)\right]^2$, is solely determined by the initial value $F(0)$.

The physics governing the escape of F and a from the periodic trap has not been discovered. If the value of the energy density $\varrho(t)$ decreases to the point where

$$\lim_{F(t) \to 0} \frac{\varrho(t)}{F(t)^2} = 0$$

then $\lim_{F(t) \to 0} \dot{F}(t) = 0$, and $F \to 0$; a is controlled by F (because $a^3 \cosh(F) = $ constant) and will escape the periodic trap and very rapidly transition to a constant $a(t_{\text{end of inflation}}) \approx a_1$, $(F(t) \equiv 0 \forall t > t_{\text{end of inflation}})$. However the governing physics may have more similarity with quantum mechanical tunneling through a barrier than the decay of the controlling parameter $\varrho(t)$ to zero.

6.1 Dark Photon

The photon field, carried by a type-2 real bosonic spinor field $\psi_2{}^a$ in this model, is mapped to the electromagnetic four-vector potential $\mathbb{A}_\alpha$ through a nonlinear (in $\mathbf{u}$) interaction with the unit field:

$$\mathbb{A}^\alpha = \frac{1}{\sqrt{\widetilde{\mathbf{u}}\,\sigma\mathbf{u}}}\,\widetilde{\mathbf{u}}\,\sigma\,\overline{\tau}^\alpha\,\Psi_2. \tag{44}$$

$\mathbb{A}^\alpha$ is a true $\overline{SO(3,1;\mathbb{R})}$ vector field since Ψ_2 is a type-2 spinor. In general, the type-2 real bosonic spinor photon field $\psi_2{}^a$ is mapped to the electromagnetic spacetime potential $\mathbb{A}_B$ through an interaction with the unit field that includes all eight components:

$$\mathbb{A}^B = (\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3, \mathbb{A}_4, \mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3, \mathbb{C}_4)_D\,\mathbb{G}^{DB} = \left(\overrightarrow{\mathbb{A}}, \mathbb{A}_4, \overrightarrow{\mathbb{C}}, \mathbb{C}_4\right)_D \mathbb{G}^{DB}$$

$$= \frac{1}{\sqrt{\widetilde{\mathbf{u}}\,\sigma\mathbf{u}}}\,\widetilde{\mathbf{u}}\,\sigma\,\overline{\tau}^B\,\Psi_2. \tag{45}$$

The $(\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3, \mathbb{C}_4)$ are $SO(3,1;\mathbb{R})$ scalars; for notational convenience, the $(\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3)$ are collected into $\overrightarrow{\mathbb{C}}$.

For the representations employed in this paper

$$\mathbb{A}^B = \frac{1}{\sqrt{2}} \begin{cases} e^{-\frac{F}{2}}\psi_2{}^3 + e^{\frac{F}{2}}\psi_2{}^7 \\ e^{-\frac{F}{2}}\psi_2{}^4 + e^{\frac{F}{2}}\psi_2{}^8 \\ -e^{-\frac{F}{2}}\psi_2{}^1 - e^{\frac{F}{2}}\psi_2{}^5 \\ e^{-\frac{F}{2}}\psi_2{}^1 - e^{\frac{F}{2}}\psi_2{}^5 \\ e^{-\frac{F}{2}}\psi_2{}^4 - e^{\frac{F}{2}}\psi_2{}^8 \\ -e^{-\frac{F}{2}}\psi_2{}^3 + e^{\frac{F}{2}}\psi_2{}^7 \\ -e^{-\frac{F}{2}}\psi_2{}^6 + e^{\frac{F}{2}}\psi_2{}^2 \\ e^{-\frac{F}{2}}\psi_2{}^6 + e^{\frac{F}{2}}\psi_2{}^2 \end{cases} \tag{46}$$

In a Minkowski limit $(F = 0)$ the electric and magnetic field frame components $\{\mathbb{E}^\mu, \mathbb{B}^\mu\}$ are given (after inflation ends) by

$$\begin{pmatrix} \overrightarrow{\mathbb{E}} \\ \mathbb{E}^4 \end{pmatrix} = \begin{pmatrix} -\frac{1}{c}\frac{\partial}{\partial t}\overrightarrow{\mathbb{A}} + \overrightarrow{\nabla}\mathbb{A}_4 + \overrightarrow{\nabla}\times\overrightarrow{\mathbb{C}} \\ 0 \end{pmatrix}. \tag{47}$$

and

$$\begin{pmatrix} \overrightarrow{\mathbb{B}} \\ \mathbb{B}^4 \end{pmatrix} = \begin{pmatrix} \overrightarrow{\nabla}\times\overrightarrow{\mathbb{A}} + \frac{1}{c}\frac{\partial}{\partial t}\overrightarrow{\mathbb{C}} + \overrightarrow{\nabla}\mathbb{C}_4 \\ -\overrightarrow{\nabla}\cdot\overrightarrow{\mathbb{C}} - \frac{1}{c}\frac{\partial}{\partial t}\mathbb{C}_4 \end{pmatrix}. \tag{48}$$

We note that imposing $\overrightarrow{\mathbb{C}} = \overrightarrow{\nabla}\mathfrak{C}$ and $\mathbb{C}_4 = -\frac{1}{c}\frac{\partial}{\partial t}\mathfrak{C}$, where $\overrightarrow{\nabla}\cdot\overrightarrow{\nabla}\mathfrak{C} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathfrak{C} =$ const $\times$ magnetic monopole charge density $= 0$ removes the $\left(\overrightarrow{\mathbb{C}}, \mathbb{C}_4\right)$ dependence from $\mathbb{E}^\mu$ and $\mathbb{B}^\mu$. If the $(\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3, \mathbb{C}_4)$ are decoupled from Standard Model degrees of freedom in this manner, then they comprise one set of **dark degrees of freedom**. These dark degrees of freedom receive energy at $t = 0$, just like all of the other degrees of freedom.

6.2 Fluctuations of the quantized unit field

It is widely believed that primordial curvature perturbations sourced by quantum field fluctuations during the early time history of the universe evolve into macroscopic perturbations whose time evolution lead to cosmological structure formation. To calculate the quantum fluctuations the bosonic spinor unit field, which satisfies nonlinear field equations, must be quantized using an approach that guarantees the renormalizability of the quantum theory and the correct normalization of the unit field. Quantization should be performed after the ϱ and p functions have been redefined and associated to a field, as in Eq.[16]. Since at comoving time t = 0 a non-vanishing average current drift velocity $\overrightarrow{J}_D$ exists in this model there will be a non-isotropic term in the total energy-momentum tensor. One needs to perturb the FLRW metric for $g_{\alpha\beta}$ simply to obtain the classical solution, then carry out quantization. The investigation is ongoing.

7 Appendix 1: Problem Formulation

Let $\mathbb{X}_{3,1}$ denote a connected orientable four-dimensional, Hausdorff, smooth pseudo-Riemannian manifold whose local tangent spaces are isomorphic to flat Minkowski spacetime $\mathbb{M}_{3,1}$. We assume that the Stiefel - Whitney class $W_2(\mathbb{X}_{3,1})$ vanishes, so that $\mathbb{X}_{3,1}$ admits a spin structure [11] [18] and spinor fields may defined on this manifold. Let **g** denote the pseudo-Riemannian metric tensor on $\mathbb{X}_{3,1}$. The signature of the metric **g** is assumed to be $(+ + +-)$ almost everywhere [because the scale factor $a(t)$ may periodically vanish]. $(\mathbb{X}_{3,1}; g)$ is called a spacetime. We assume that $\mathbf{g} \leftrightarrow g_{\alpha\beta} = g_{\alpha\beta}(x^\mu)$ carries the Newton-Einstein gravitational degrees of freedom. Here Greek indices run from 1 to 4, and x^μ are local coordinates on $\mathbb{X}_{3,1}$. We are using the usual component notation in local charts. The tensor covariant derivative with respect to the symmetric connection associated with the metric g will be denoted by a semicolon. A partial derivative is sometimes denoted by a comma. When enclosed in parenthesis Greek indices denote flat Minkowski $SO(3,1)$ frame indices; otherwise Greek indices are $\mathbb{X}_{3,1}$ vector indices. Let $x^\alpha = (x, y, z, c\,t)$ be local comoving coordinates for a neighborhood of $p \in \mathbb{X}_{3,1}$. Following [5] [16] we introduce an orthonormal tetrad $E_\alpha^{(\mu)}$ (with inverse $E_{(\mu)}^\alpha$) that defines a set of four linearly independent vector fields such that $\mathbb{E} \equiv det\left(E_\alpha^{(\mu)}\right) = \sqrt{-g} \neq 0$. The components are assumed to be orthonormal: $g_{\alpha\beta}\, E_{(\mu)}^\alpha\, E_{(\nu)}^\beta = \eta_{(\mu)(\nu)} = \mathrm{diag}(+1, +1, +1, -1)$ with respect to the $\mathbb{X}_{3,1}$ metric $g_{\alpha\beta}$.

In this paper we study a universe that is modeled in terms of a trivial fiber bundle $\mathbb{X}_{3,1} \times \mathbb{R}^{4,4}$ with base space $\mathbb{X}_{3,1}$ and fiber $\mathbb{R}^{4,4}$, with the trivial projection onto the first factor. $\mathbb{R}^{4,4}$ denotes a real, eight dimensional pseudo-Euclidean vector space over $\mathbb{R}$. Let $q \in \mathbb{R}^{4,4}$ and $q^A \in \mathbb{R}$, $A, B, ... = 1, 2, ..., 8$, denote the Cartesian coordinates of q with respect to a definite, right-handed

orthonormal frame. $O(4, 4; \mathbb{R})$ (respectively, $SO(4, 4; \mathbb{R})$) may be defined as the group of all real matrices (respectively, with unit determinant) that preserve the quadratic form

$$(q^8)^2 + (q^1)^2 + (q^2)^2 + (q^3)^2 - \left[(q^4)^2 + (q^5)^2 + (q^6)^2 + (q^7)^2 \right].$$

$O(4, 4; \mathbb{R})$ is a pseudo-orthogonal Lie group that possess two connected components [3],[12], with $SO(4, 4; \mathbb{R})$ being the identity component (the connected component containing the identity matrix). $\mathrm{Spin}(4, 4, \mathbb{R})$, alternatively denoted $\overline{SO(4, 4; \mathbb{R})}$, is the 2-to-1 covering group of $SO(4, 4; \mathbb{R})$. $\mathbb{R}^{4,4}$ may be endowed with both $SO(4, 4; \mathbb{R})$-invariant and $\overline{SO(4, 4; \mathbb{R})}$-invariant pseudo-Euclidean metrics.

The $SO(4, 4; \mathbb{R})$-invariant pseudo-Euclidean metric tensor $\mathbb{G}$ (respectively, inverse $\mathbb{G}^{-1}$) may be defined in terms of its components $\mathbb{G}_{AB}$ (respectively, $\left(\mathbb{G}^{-1}\right)^{AB} = \mathbb{G}^{AB}$) that, with respect to a definite Cartesian, right-handed orthonormal frame, are numerically equal to

$$\mathbb{G}_{AB} = \mathbb{G}^{AB} = \begin{pmatrix} \eta_{3,1} & 0 \\ 0 & -\eta_{3,1} \end{pmatrix} \tag{49}$$

where $\eta_{3,1} = \mathrm{diag}(1, 1, 1, -1)$ is the pseudo-Euclidean metric on flat four-dimensional Minkowski space-time $M_{3,1}$. The indefinite inner product is realized as $\mathbb{R}^{4,4} \times \mathbb{R}^{4,4} \ni (a, b) \mapsto < a, b > = \mathbb{G}_{AB} a^A b^B \in \mathbb{R}$. Here A, B, ... = 1, ... , 8 may be regarded as $\mathbb{R}^{4,4}$ vector indices.

The fiber $\mathbb{R}^{4,4}$ also admits a $\overline{SO(4, 4; \mathbb{R})}$-invariant pseudo-Euclidean 8×8 spinor metric σ,

$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{50}$$

Given spinors $\psi, \chi \in \mathbb{R}^{4,4}$, $\widetilde{\psi} \sigma \chi$ defines a $\overline{SO(4, 4; \mathbb{R})}$-invariant indefinite inner product on $\mathbb{R}^{4,4}$. Here, following Dirac, $\widetilde{\psi}$ denotes the transpose of ψ.

Note that σ^2 is equal to the unit matrix, so that the eigenvalues of σ of are ± 1. Since the trace of σ is zero, these eigenvalues occur with equal multiplicity. The matrix elements of σ are denoted as σ_{ab}, $a, b, ... = 1, ...8$. Generally, the early lower case Latin character indices $a, b, ...$ denote $\mathrm{Spin}(4, 4, \mathbb{R}) \equiv \overline{SO(4, 4; \mathbb{R})}$ spinor indices, and run from 1 to 8. A spinor element $\psi \in \mathbb{R}^{4,4}$ is defined to have components $\psi^a \in \mathbb{R}$. The "left-handed" and "right-handed" $\mathrm{Spin}(8)$ spinors have $\overline{S0(4, 4; \mathbb{R})}$ counterparts that are denoted ψ_1 and ψ_2 in this paper, and transform, respectively, under two inequivalent real 8×8 irreducible spinor representations of $\overline{S0(4, 4; \mathbb{R})}$ that we call $D_{(1)}$ (type 1) and $D_{(2)}$ (type 2). For simplicity, we do not distinguish the spinor index on ψ_2 from that on ψ_1 using a convention such as ψ_{1a} and $\psi_2{}^{\dot{a}}$ for spinor components. As long as no confusion arises, this simplified notation may be justified.

When applying this formalism to the physical world we usually restrict the allowed transformations as follows:

7.0.1 Restriction of $\overline{S0(4,4;\mathbb{R})}$ to $\overline{S0(3,1;\mathbb{R})}$

Let $(x^1, x^2, x^3, x^4, x^5, x^6, x^7, x^8) \in \mathbb{R}^{4,4}$. In this paper we assume that for this model to possess physical significance the action of $\overline{S0(4,4;\mathbb{R})}$ must be restricted to $\overline{S0(3,1;\mathbb{R})}$ in a manner that links transformations of x^5, x^6, x^7, x^8 to x^1, x^2, x^3, x^4. Therefore we are led to identify $\mathbb{R}^{4,4}$ with the single particle relativistic phase space $M_{3,1} \oplus {}^*M_{3,1}$, $\mathbb{R}^{4,4} \cong M_{3,1} \oplus {}^*M_{3,1}$, and to restrict $\overline{S0(4,4;\mathbb{R})}$ to the subgroup

$$\mathbb{K} = \left\{ g \in \overline{S0(4,4;\mathbb{R})} \mid g = \begin{pmatrix} h & 0 \\ 0 & \widetilde{h}^{-1} \end{pmatrix}, \; h \in \overline{S0(3,1;\mathbb{R})} \right\}. \tag{51}$$

Under the action of $L \in S0(3,1;\mathbb{R})$, $x^\alpha \mapsto \overline{x}^\alpha = L^\alpha{}_\beta x^\beta$ and $x^{4+\alpha} \equiv p_\alpha \mapsto \overline{p}_\alpha = p_\beta L_\alpha{}^\beta$; $\overline{p}_\alpha \overline{x}^\alpha = p_\alpha x^\alpha$ is an invariant, since for $g \in \mathbb{K}$

$$M_{3,1} \oplus {}^*M_{3,1} \ni \begin{pmatrix} x \\ \widetilde{p} \end{pmatrix} \mapsto g \begin{pmatrix} x \\ \widetilde{p} \end{pmatrix} = \begin{pmatrix} hx \\ \widetilde{h}^{-1}\widetilde{p} \end{pmatrix} = \begin{pmatrix} hx \\ \widetilde{p\,h^{-1}} \end{pmatrix} \tag{52}$$

7.1 Preferred orthonormal frame $\mathfrak{F}_A^a$ on $\mathbb{R}^{4,4}$; split octonion algebra

Let $\mathbf{O}_S$ denote the split octonion algebra. The new "unit field" interaction is carried by a distinguished split octonion $\mathbf{u} \in \mathbf{O}_S$ that is algebraically distinguished by the fact that $\mathbf{u}$ is the $\mathbf{O}_S$ ***multiplicative identity*** element [21]. The components of the *unit* field $\mathbf{u}$ with respect to a (non-associative) canonical oriented spinor basis $\mathbf{e}_a$ of the split octonion algebra are denoted as u^a. Throughout this paper we work with the components u^a of $\mathbf{u}$, rather than with $\mathbf{u}$, to avoid the complexities of the non-associativity of the split octonion algebra. In this paper we seek a solution u^a to the coupled field equations of the form $u^a = \frac{1}{\sqrt{2}}\left(0, e^{-\frac{F(t)}{2}}, 0, 0, 0, e^{\frac{F(t)}{2}}, 0, 0\right)$.

We do not introduce a characteristic mass for the $\mathbf{u}$ field quanta, and hence we regard the $\mathbf{u}$ unit field to be dimensionless. The real unit spinor field $u^a = u^a(x^\mu)$ carries the massless "unit field" interaction and defines a preferred orthonormal frame $\mathfrak{F}_A^a$ on $\mathbb{R}^{4,4}$. $\mathfrak{F}_A^a$ is defined in terms of $\mathbf{u}$ and natural extension of the Dirac matrices as follows. Define the tau matrices by requiring that the tau matrices satisfy (the tilde denotes transpose)

$$\sigma\overline{\tau}^A = \widetilde{\sigma\tau^A} = \widetilde{\tau}^A\sigma \tag{53}$$

and

$$\tau^A\overline{\tau}^B + \tau^B\overline{\tau}^A = 2\,\mathbb{I}_{8\times8}\,\mathbb{G}^{AB} = \overline{\tau}^A\tau^B + \overline{\tau}^B\tau^A, \tag{54}$$

where $\mathbb{I}_{8\times8}$ denotes the 8×8 unit matrix. Denoting the matrix elements of τ^A by $\tau^{A\,a}{}_b$, we may write Eq.[53] as

$$\overline{\tau}^A{}_{ab} = \tau^A{}_{ba} \tag{55}$$

where we have used σ to lower the spinor indices. The tau matrices verify an important identity [19] : Let M be any 8×8 matrix satisfying

$$\widetilde{\sigma M} = \sigma M \tag{56}$$

and moreover transforming under $\overline{S0(4,4;\mathbb{R})}$ according to

$$M \mapsto D_{(1)} M D_{(1)}{}^{-1} \tag{57}$$

Then

$$\tau_A M \overline{\tau}^A = \mathbb{I}_{8 \times 8} \, tr(M) \tag{58}$$

($\mathbb{I}_{8 \times 8}$ denotes the 8×8 unit matrix).

7.1.1 Components of $\mathfrak{F}_A^a$

The components $\mathfrak{F}_A^a$ of the $\mathbb{R}^{4,4}$ orthogonal frame are

$$\mathfrak{F}_A^a = \frac{1}{\sqrt{\widetilde{\mathbf{u}} \, \sigma \, \mathbf{u}}} \, \tau_A {}^a{}_b \, u^b \tag{59}$$

and its inverse is

$$\mathfrak{F}_a^A = \frac{1}{\sqrt{\widetilde{\mathbf{u}} \, \sigma \, \mathbf{u}}} \, u^c \sigma_{cb} \overline{\tau}^{A\,b}{}_a \tag{60}$$

Using Eq.[58] one easily verifies that

$$\{\mathbb{I}_{8 \times 8}\}_b^a = \delta_b^a = \mathfrak{F}_A^a \, \mathfrak{F}_b^A, \tag{61}$$

and since a matrix commutes with its inverse,

$$\delta_B^A = \mathfrak{F}_a^A \, \mathfrak{F}_B^a. \tag{62}$$

Why is this important? The orthonormal frame $\mathfrak{F}_A^a$ may be employed to map a vector field associated to a fundamental Boson "particle" to a spinor field, and then back again.

7.1.2 Split octonion algebra

A nonassociative alternative multiplication of the oriented spinor basis e_a (respectively, oriented vector basis ϵ_A) may be defined that endows the real vector space $\mathbb{R}^{4,4}$ with the structure of a normed algebra with multiplicative unit. This is accomplished by specifying the multiplication constants m_{ab}^c (respectively, m_{AB}^C) of the algebra, which verify

$$\begin{aligned} e_a \, e_b &= e_c \, m_{ab}^c \\ \epsilon_A \, \epsilon_B &= \epsilon_C \, m_{AB}^C \end{aligned} \tag{63}$$

The set of m_{ab}^c (respectively, m_{AB}^C) defined by

$$\begin{aligned} m_{ab}^c &= \mathfrak{F}_a^A \, \tau_A {}^c{}_b \\ m_{AB}^C &= \mathfrak{F}_c^C \, \tau_A {}^c{}_b \, \mathfrak{F}_B^b \end{aligned} \tag{64}$$
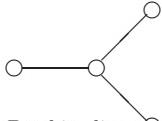
Fig. 1 Dynkin diagram for D_4

is a realization of the multiplication constants of the algebra. It has been proven that the nonassociative product defined by Eq.[63] and Eq.[64] of the spinor basis e_a (respectively, of the vector basis ϵ_A) endows the real vector space $\mathbb{R}^{4,4}$ with the structure of the split octonion algebra over the reals [20]. The multiplicative unit element of this split octonion algebra is

$$id = e_a u^a = \epsilon_8 = \mathbf{u}. \tag{65}$$

8 Appendix 2: Spinor representations of $\overline{SO(4,4;\mathbb{R})}$ on $\mathbb{R}^{4,4}$

8.1 Representations of SO(8, $\mathbb{C}$)

Both $O(4,4;\mathbb{R})$ and $SO(4,4;\mathbb{R})$ are real forms of the classical complex orthogonal group $O(8,\mathbb{C})$. The relationship between Clifford algebras C_n and the spinor representations of the classical complex orthogonal groups is well known. Summarizing a few basic facts that may be found in, for example, Boerner, *The Representations of Groups* [3], the Clifford algebra C_8 may be defined as the algebra generated by a set of 8 elements $e_j, j = 1, \ldots, 8$, that anticommute with each other and have unit square $e_j e_k + e_k e_j = 2\delta_{jk}\mathbb{I}_{16\times 16}$, $\mathbb{I}$ = unit matrix. The scaled commutators $\frac{1}{4}(e_j e_k - e_k e_j)$ computed from the 16-dimensional irreducible representation of the e_j are the infinitesimal generators of a reducible 16-dimensional representation of Spin(8, $\mathbb{C}$), which is the universal double covering of the special orthogonal group SO(8, $\mathbb{C}$). This 16-dimensional representation of is fully reducible to the direct sum of two inequivalent irreducible 8×8 spin representations of the infinitesimal generators of Spin(8, $\mathbb{C}$), which leads to the identification of type 1 and type 2 spinors. The fundamental irreducible vector representation of SO(8, $\mathbb{C}$) is also 8×8. The Dynkin diagram for $D_4 \cong$ SO(8) is symmetrical and pictured in Figure 1. The three outer nodes correspond to the vector representation (left-most node), type 1 spinor and type 2 spinor representations of Spin(8), and the central node corresponds to the adjoint representation.

As is the case with $SO(2n;\mathbb{C})$, the basic spinor representation of the pseudo-orthogonal group $SO(4,4;\mathbb{R})$ may be constructed from the irreducible generators t^A, $A = 1,...,8$, of the pseudo-Clifford algebra $C_{4,4}$ [3],[4], [15]. Lord [15] has indicated a general procedure for constructing the spinor representations of $SO(2n;\mathbb{C})$ of the first and second kind (handedness) from the generators of C_{2n-2}. Following Lord we shall call such irreducible C_{2n-2} generators "reduced Brauer-Weyl generators" [4] . Using Lord's procedure we construct the representation of $\overline{S0(4,4;\mathbb{R})}$ by first defining the real 8×8 matrix reduced

Brauer-Weyl generators $\tau^A, \overline{\tau}^A$, $A, B, \ldots = 1, \ldots, 8$, of the pseudo-Clifford algebra $C_{4,4}$ that anticommute and have square ± 1 according to Eq.[53] and Eq.[54]. We adopt a real irreducible 8×8 matrix representation of the tau matrices (see the Appendix) in which $\overline{\tau}^8 = \mathbb{I}_{8\times8} = \tau^8$. Then by Eq.[54] $\overline{\tau}^A = -\tau^A$ for A = 1, ... ,7. Hence, again by Eq.[54], $(\tau^A)^2$ is equal to $-\mathbb{I}_{8\times8}$ for A = 1,2,3 and is equal to $+\mathbb{I}_{8\times8}$ for A = 4,5,6, 7, 8.

8.2 Transformation under action of $\overline{S0(4,4;\mathbb{R})}$

The special Lorentz transformation properties of the theory may be determined by constructing a real reducible 16×16 matrix representation of $\overline{S0(4,4;\mathbb{R})}$ utilizing the irreducible generators t^A, $A = 1, \ldots, 8$ of the (pseudo-) Clifford algebra $C_{4,4}$. Continuing to employ Lord's general procedure [15] we define the irreducible generators t^A as

$$t^A = \begin{pmatrix} 0 & \overline{\tau}^A \\ \tau^A & 0 \end{pmatrix}. \tag{66}$$

Let $g \in \overline{S0(4,4;\mathbb{R})}$. The 16×16 basic spinor representation of $\overline{S0(4,4;\mathbb{R})}$ is reducible into the two real 8×8 inequivalent irreducible spinor representations $D_{(1)}(g)$ and $D_{(2)}(g)$ of $\overline{S0(4,4;\mathbb{R})}$. The reduced generators of the two real 8×8 spinor representations $D_{(1)}(g)$ and $D_{(2)}(g)$ of $\overline{S0(4,4;\mathbb{R})}$ follow from the calculation of the infinitesimal generators

$$t^A t^B - t^B t^A = \begin{pmatrix} \overline{\tau}^A \tau^B - \overline{\tau}^B \tau^A & 0 \\ 0 & \tau^A \overline{\tau}^B - \tau^B \overline{\tau}^A \end{pmatrix}$$
$$= 4 \begin{pmatrix} D_{(1)}{}^{AB} & 0 \\ 0 & D_{(2)}{}^{AB} \end{pmatrix}, \tag{67}$$

of the 16-component spinor representation of $\overline{S0(4,4;\mathbb{R})}$. We see that, as is in fact well known from the general theory, the 16-component spinor representation of $\overline{S0(4,4;\mathbb{R})}$ is the direct sum of two (inequivalent) real 8×8 irreducible spinor representations $D_{(1)} = D_{(1)}(g)$ and $D_{(2)} = D_{(2)}(g)$ of $\overline{S0(4,4;\mathbb{R})} \ni g$ that are generated by $D_{(1)}{}^{AB}$ and $D_{(2)}{}^{AB}$ respectively. This we record as

$$4 D_{(1)}{}^{AB} = \overline{\tau}^A \tau^B - \overline{\tau}^B \tau^A \tag{68}$$

and

$$4 D_{(2)}{}^{AB} = \tau^A \overline{\tau}^B - \tau^B \overline{\tau}^A \tag{69}$$

For completeness we remark that the generators of the two spinor types are images of the projection operators

$$\chi_{\pm} = \frac{1}{2}(1 \pm t^9)$$
$$\chi_{+} = \begin{pmatrix} \mathbb{I}_{8\times8} & 0 \\ 0 & 0 \end{pmatrix}$$
$$\chi_{-} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{I}_{8\times8} \end{pmatrix}, \tag{70}$$

where

$$t^9 = t^1 t^2 t^3 t^4 t^5 t^6 t^7 t^8 = \begin{pmatrix} \overline{\tau}^0 & 0 \\ 0 & \tau^0 \end{pmatrix}. \tag{71}$$

Here

$$\overline{\tau}^0 = \overline{\tau}^1 \tau^2 \overline{\tau}^3 \tau^4 \overline{\tau}^5 \tau^6 \overline{\tau}^7 = \tau^1 \tau^2 \tau^3 \tau^4 \tau^5 \tau^6 \tau^7 \tag{72}$$

and

$$\tau^0 = \tau^1 \, \overline{\tau}^2 \, \tau^3 \, \overline{\tau}^4 \, \tau^5 \, \overline{\tau}^6 \, \tau^7 = -\tau^1 \tau^2 \tau^3 \tau^4 \tau^5 \tau^6 \tau^7 = -\overline{\tau}^0 \tag{73}$$

The representation of the tau matrices is irreducible. $\overline{\tau}^0$ has square equal to $+\mathbb{I}_{8\times 8}$ and commutes with each of the τ^A matrices (and therefore with all of their products). Therefore we conclude that $\overline{\tau}^0 = \pm\mathbb{I}_{8\times 8}$ in any irreducible representation. $\overline{\tau}^0 = \mathbb{I}_{8\times 8}$ in the irreducible representation given in the Appendix.

Let $\omega_{AB} = -\omega_{BA} \in \mathbb{R}$, $A, B = 1, \ldots, 8$, enumerate a set of 28 real parameters that coordinatize $g = g(\omega) \in \overline{S0(4,4;\mathbb{R})}$. Also, let $L = L(g) \in SO(4,4;\mathbb{R})$ have matrix elements $L^A{}_B$, $\omega^\sharp$ denote the real 8×8 matrix with matrix elements $\omega^A{}_B = \mathbb{G}^{AC}\omega_{CB}$, $\omega_1 = \frac{1}{2}\omega_{AB}D_{(1)}{}^{AB}$ and $\omega_2 = \frac{1}{2}\omega_{AB}D_{(2)}{}^{AB}$. We find that

$$D_{(1)} = D_{(1)}(g) = \exp\left(\frac{1}{2}\omega_1\right)$$

$$D_{(2)} = D_{(2)}(g) = \exp\left(\frac{1}{2}\omega_2\right)$$

$$L^A{}_B = L^A{}_B(g) = \left\{\exp\left(\omega^\sharp\right)\right\}^A{}_B \tag{74}$$

where, under the action of $\overline{S0(4,4;\mathbb{R})}$,

$$\widetilde{D_{(1)}}{}^{AB}\sigma = -\sigma D_{(1)}{}^{AB} \Rightarrow \widetilde{D}_{(1)}\sigma = \sigma D_{(1)}{}^{-1} \tag{75}$$

$$\widetilde{D_{(2)}}{}^{AB}\sigma = -\sigma D_{(2)}{}^{AB} \Rightarrow \widetilde{D}_{(2)}\sigma = \sigma D_{(2)}{}^{-1} \tag{76}$$

$$L^A{}_C \mathbb{G}_{AB} L^B{}_D = \mathbb{G}_{CD} = \left\{\widetilde{L}\mathbb{G}L\right\}_{CD} \tag{77}$$

$$L^A{}_B \overline{\tau}^B = D_{(1)}{}^{-1} \overline{\tau}^A D_{(2)} \tag{78}$$

$$L^A{}_B \tau^B = D_{(2)}{}^{-1} \tau^A D_{(1)} \tag{79}$$

The canonical 2-1 homomorphism $\overline{S0(4,4;\mathbb{R})} \to S0(4,4;\mathbb{R}) : g \mapsto L(g)$ is given by

$$8 \, L^A{}_B = tr\left(D_{(1)}{}^{-1} \overline{\tau}^A D_{(2)} \, \tau^C\right) \mathbb{G}_{CB}, \tag{80}$$

where tr denotes the trace. Note that $D_{(1)}(g(\omega)) = D_{(2)}(g(\omega))$ when $\omega_{A8} = 0$, i.e., when one restricts $\overline{S0(4,4;\mathbb{R})}$ to

$$\overline{S0(3,4;\mathbb{R})} = \left\{g \in \overline{S0(4,4;\mathbb{R})} \ \middle| \right.$$

$$g = \begin{pmatrix} \exp\left(\frac{1}{4}\omega_{AB}D_{(1)}{}^{AB}\right) & 0 \\ 0 & \exp\left(\frac{1}{4}\omega_{AB}D_{(2)}{}^{AB}\right) \end{pmatrix} \ \text{and} \ \omega_{A8} = 0 \right\} \tag{81}$$

[one of the real forms of $\mathrm{Spin}(7,\mathbb{C})$]. When stating mathematical spinor relationships we almost always restrict $\overline{SO(4,4;\mathbb{R})}$ to $\overline{SO(3,4;\mathbb{R})}$ without explicit notice.

We adopt a real irreducible 8×8 matrix representation of the tau matrices (see Appendix 10) in which $\overline{\tau}^8 = \mathbb{I}_{8\times 8} = \tau^8$. Then by Eq.[54] $\overline{\tau}^A = -\tau^A$ for A = 1, ... ,7. Hence, again by Eq.[54], $(\tau^A)^2$ is equal to $-\mathbb{I}_{8\times 8}$ for A = 1,2,3 and is equal to $+\mathbb{I}_{8\times 8}$ for A = 4,5,6, 7, 8.

8.3 $\overline{S0(4,4;\mathbb{R})}$ covariant multiplications

Let V_1, V_2, and V_3 be vector spaces over $\mathbb{R}$. A duality is a nondegenerate bilinear map $V_1 \times V_2 \to \mathbb{R}$. A triality is a nondegenerate trilinear map $V_1 \times V_2 \times V_3 \to \mathbb{R}$. A triality may be associated with a bilinear map that some authors call a "multiplication" [2] by dualizing, $V_1 \times V_2 \to {}^*V_3 \cong V_3$. There are two especially simple multiplications that possess covariant transformation laws under the action of $\overline{S0(4,4;\mathbb{R})}$. Assume that $u \mapsto \overline{u} = D_{(1)}\, u$, $\psi_1 \mapsto \overline{\psi}_1 = D_{(1)}\, \psi_1$ and $\psi_2 \mapsto \overline{\psi}_2 = D_{(2)}\, \psi_2$ under the action of $\overline{S0(4,4;\mathbb{R})}$. The first multiplication $m_1{}^A : \mathbb{R}^{4,4} \times \mathbb{R}^{4,4} \to \mathbb{R}^{4,4}$ is defined by

$$Q^A = \widetilde{u}\, \sigma\, \overline{\tau}^A\, \psi_2. \tag{82}$$

For fixed u^a, $Q^A \in \mathbb{R}^{4,4}$ depends on 8 real parameters arranged into the type-2 spinor ψ_2.

The second multiplication $m_2{}^{AB} : \mathbb{R}^{4,4} \times \mathbb{R}^{4,4} \to V_3$ has an image in $V_3 \cong \mathbb{R}^{4,4} \times \mathbb{R}^{4,4}$, and depends on 8 real parameters (for fixed u^a) arranged into the type-1 spinor ψ_1:

$$Q^{AB} = \widetilde{u}\, \sigma\, \overline{\tau}^A\, \tau^B\, \psi_1. \tag{83}$$

We emphasize that, for fixed u, Q^{AB} possesses only 8 degrees of freedom corresponding to the 8 independent degrees of freedom of ψ_1, so we also refer to this map as a "multiplication." Eq.[83] may be solved for the components $\psi_1{}^a = \psi_1{}^a(Q^{AB})$.

Suppose we break the $\overline{S0(4,4;\mathbb{R})}$ symmetry down to $\overline{S0(3,1;\mathbb{R})}$ according to paragraph 7.0.1. We note that $\mathbb{G}$ is invariant under $\overline{S0(3,1;\mathbb{R})}$ since it is invariant under $\overline{S0(4,4;\mathbb{R})}$. It is convenient to define a $S0(3,1;\mathbb{R})$-invariant symplectic structure Ω on $\mathbb{R}^{4,4}$ (and a complex structure on the split octonion algebra) by

$$\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{84}$$

where 0 denotes the 4 x 4 zero matrix and 1 denotes the 4 x 4 unit matrix.

Q^{AB} may be expressed in terms of a linear combination of 'simple' objects that transform as $M_{3,1}$ tensors of appropriate rank. Consider first two arbitrary $M_{3,1}$ vectors with components $\mathbb{A}_\mu$ and $\mathbb{B}_\mu$. Q^{AB} may be written as a linear combination of $\mathbb{A}^\alpha\, \mathbb{B}^\beta \pm \mathbb{A}^\beta\, \mathbb{B}^\alpha$, $\epsilon^{\alpha\beta\mu\nu}\mathbb{A}_\mu\, \mathbb{B}_\nu$ and $\eta^{\alpha\beta}\, \eta^{\mu\nu}\mathbb{A}_\mu\, \mathbb{B}_\nu$, but this is does

not yield a linear relationship between Q^{AB} and $\{\mathbb{A}_\mu, \mathbb{B}_\mu\}$. Also, since Q^{AB} has exactly eight independent degrees of freedom it cannot be expressed in terms of a traceless symmetric $M_{3,1}$ rank 2 tensor, which has nine independent components. However we find that the Q^{AB} may be represented in terms of an arbitrary antisymmetric $M_{3,1}$ rank 2 tensor $\mathbb{F}^{\beta\alpha} = -\mathbb{F}^{\alpha\beta}$ and two $S0(3,1;\mathbb{R})$ scalars $q_4 = B_4$ and $q_8 = E_4$ according to

$$Q^{AB} = \begin{pmatrix} \mathbb{F}^{\alpha\,\beta} & {}^*\mathbb{F}^\alpha{}_\beta \\ {}^*\mathbb{F}_\alpha{}^\beta & \mathbb{F}_{\alpha\,\beta} \end{pmatrix} + q_4 \Omega^{AB} + q_8 \mathbb{G}^{AB}, \tag{85}$$

where ${}^*\mathbb{F}^{\alpha\,\beta}$ is dual to $\mathbb{F}^{\alpha\,\beta}$ and defined by ${}^*\mathbb{F}^{\mu\nu} = -\frac{1}{2}\epsilon^{\alpha\beta\mu\nu}\mathbb{F}_{\alpha\,\beta}$. Note that $Q^{[AB]} = \frac{1}{2}\left(Q^{AB} - Q^{BA}\right)$ is independent of q_8, that q_8 may be expressed as Q^{88}, and that q_4 may be expressed as Q^{48} .

If we employ the representation of the tau matrices given in the Appendix, and then solve for the $\psi_1{}^a$ for the case $u = \frac{1}{\sqrt{2}}(0,1,0,0,0,1,0,0)$ then we obtain

$$\psi_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -B_3 + B_4 \\ E_3 + E_4 \\ B_1 - E_2 \\ B_2 + E_1 \\ -B_3 - B_4 \\ -E_3 + E_4 \\ B_1 + E_2 \\ B_2 - E_1 \end{pmatrix} \tag{86}$$

9 Appendix 3: Spin Connection

Recall that Greek indices run from 1 to 4. When enclosed in parenthesis they are flat Minkowski spacetime $SO(3,1)$ indices; otherwise they are $\mathbb{X}_{3,1}$ vector indices. Thee metric tensor $g_{\alpha\beta}$ has signature $(+++-)$. Let $x^\alpha = (x,y,z,t)$ be local comoving coordinates for a neighborhood of $p \in \mathbb{X}_{3,1}$.

Following [5] [16] we introduce an orthonormal tetrad $E_\alpha^{(\mu)}$ (with inverse $E_{(\mu)}^\alpha$) that defines a set of four linearly independent vector fields such that $\mathbb{E} \equiv det\left(E_\alpha^{(\mu)}\right) = \sqrt{-g} \neq 0$. The components are assumed to be orthonormal: $g_{\alpha\beta}\, E_{(\mu)}^\alpha\, E_{(\nu)}^\beta = \eta_{(\mu)(\nu)}$ with respect to the $\mathbb{X}_{3,1}$ metric $g_{\alpha\beta}$.

Let

$$\gamma^\alpha = E_{(\mu)}^\alpha\, \tau^{(\mu)} \tau^7. \tag{87}$$

The Lorentz connection $\omega_\mu^{(\alpha)(\beta)}$ and spin connection $\varGamma_\mu = \frac{1}{8}\omega_{\mu(\alpha)(\beta)}\left[\overline{\gamma}^{(\alpha)}\gamma^{(\beta)} - \overline{\gamma}^{(\beta)}\gamma^{(\alpha)}\right]$ are defined [16] using

$$-2\,\omega_\mu^{(\alpha)(\beta)} = E^{(\alpha)\,\kappa}\left(\partial_\mu E_\kappa^{(\beta)} - \partial_\kappa E_\mu^{(\beta)}\right) - E^{(\beta)\,\kappa}\left(\partial_\mu E_\kappa^{(\alpha)} - \partial_\kappa E_\mu^{(\alpha)}\right)$$
$$+ E^{(\alpha)\,\kappa}E^{(\beta)\,\sigma}\left(\partial_\sigma E_\kappa^{(\lambda)} - \partial_\kappa E_\sigma^{(\lambda)}\right) E_{(\lambda)\mu}. \tag{88}$$

Note that the covariant derivative $\nabla_\beta \gamma^\alpha = \gamma^\alpha{}_{;\beta} = \partial_\beta \gamma^\alpha + \left\{{}^{\ \alpha}_{\nu\ \beta}\right\} \tau^\nu + \gamma^\alpha\, \Gamma_\beta - \Gamma_\beta\, \gamma^\alpha \equiv 0$. Here $\left\{{}^{\ \alpha}_{\nu\ \beta}\right\}$ is the Christoffel symbol of the second kind. For brevity we write $\gamma^\alpha \nabla_\alpha u = \gamma^\alpha\, [u_{,\alpha} - \Gamma_\alpha u] = \gamma^\alpha\, u_{|\alpha}$.

10 Appendix 4: Irreducible Representation of the τ Matrices

The particular irreducible representation of the tau matrices employed in the calculations is

$$
\tau^1 = \begin{pmatrix}
0&0&0&0&0&0&0&-1\\
0&0&0&0&0&0&-1&0\\
0&0&0&0&0&1&0&0\\
0&0&0&0&1&0&0&0\\
0&0&0&-1&0&0&0&0\\
0&0&-1&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0
\end{pmatrix}
\quad
\tau^2 = \begin{pmatrix}
0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&-1\\
0&0&0&0&-1&0&0&0\\
0&0&0&0&0&1&0&0\\
0&0&1&0&0&0&0&0\\
0&0&0&-1&0&0&0&0\\
-1&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0
\end{pmatrix}
$$

$$
\tau^3 = \begin{pmatrix}
0&0&0&0&0&-1&0&0\\
0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&-1\\
0&0&0&0&0&0&1&0\\
0&-1&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0\\
0&0&0&-1&0&0&0&0\\
0&0&1&0&0&0&0&0
\end{pmatrix}
\quad
\tau^4 = \begin{pmatrix}
0&0&0&0&0&-1&0&0\\
0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&-1&0\\
0&1&0&0&0&0&0&0\\
-1&0&0&0&0&0&0&0\\
0&0&0&-1&0&0&0&0\\
0&0&1&0&0&0&0&0
\end{pmatrix}
$$

$$
\tau^5 = \begin{pmatrix}
0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1\\
0&0&0&0&-1&0&0&0\\
0&0&0&0&0&-1&0&0\\
0&0&-1&0&0&0&0&0\\
0&0&0&-1&0&0&0&0\\
1&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0
\end{pmatrix}
\quad
\tau^6 = \begin{pmatrix}
0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&-1&0\\
0&0&0&0&0&1&0&0\\
0&0&0&0&-1&0&0&0\\
0&0&0&-1&0&0&0&0\\
0&0&1&0&0&0&0&0\\
0&-1&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0
\end{pmatrix}
$$

$$
\tau^7 = \begin{pmatrix}
-1&0&0&0&0&0&0&0\\
0&-1&0&0&0&0&0&0\\
0&0&-1&0&0&0&0&0\\
0&0&0&-1&0&0&0&0\\
0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1
\end{pmatrix}
\quad
\tau^8 = \begin{pmatrix}
1&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1
\end{pmatrix}
$$

$$\tag{89}$$

$$|t| \le \frac{2\left(\sqrt[4]{\cosh(F_i)} + 1\right)}{\sqrt{3}\ \sqrt{\rho_i}}$$

$$\text{Singularities at } t = \pm \frac{2\sqrt[4]{\cosh(F_i)}}{\sqrt{3}\ \sqrt{\rho_i}}$$

$$\text{Possible initial states at } t = \pm \frac{2\left(\sqrt[4]{\cosh(F_i)} \pm 1\right)}{\sqrt{3}\ \sqrt{\rho_i}}$$

Fig. 2 $F(t)$ (blue) F_i (red, dashed) [limiting case of no Fermions and zero electromagnetic fields]. [Color online]

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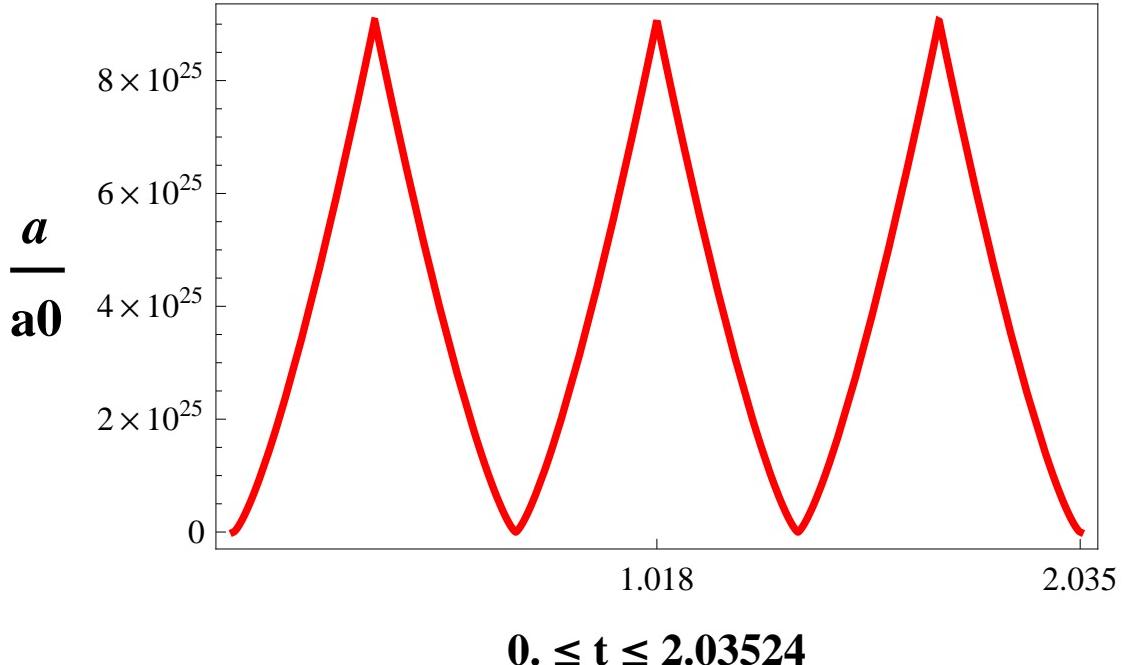
Fig. 3 Full problem: Scale factor $\frac{a(t)}{a(0)}$. [Color online]

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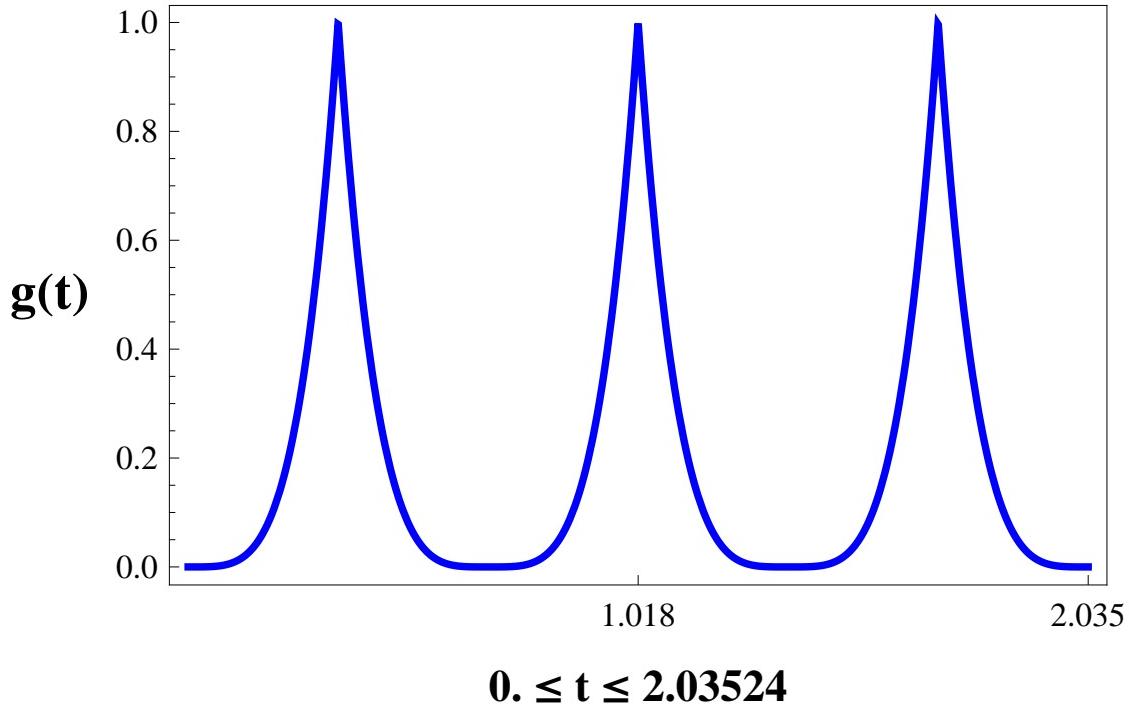
Fig. 4 Full problem: unit field $g(t)$. [Color online]

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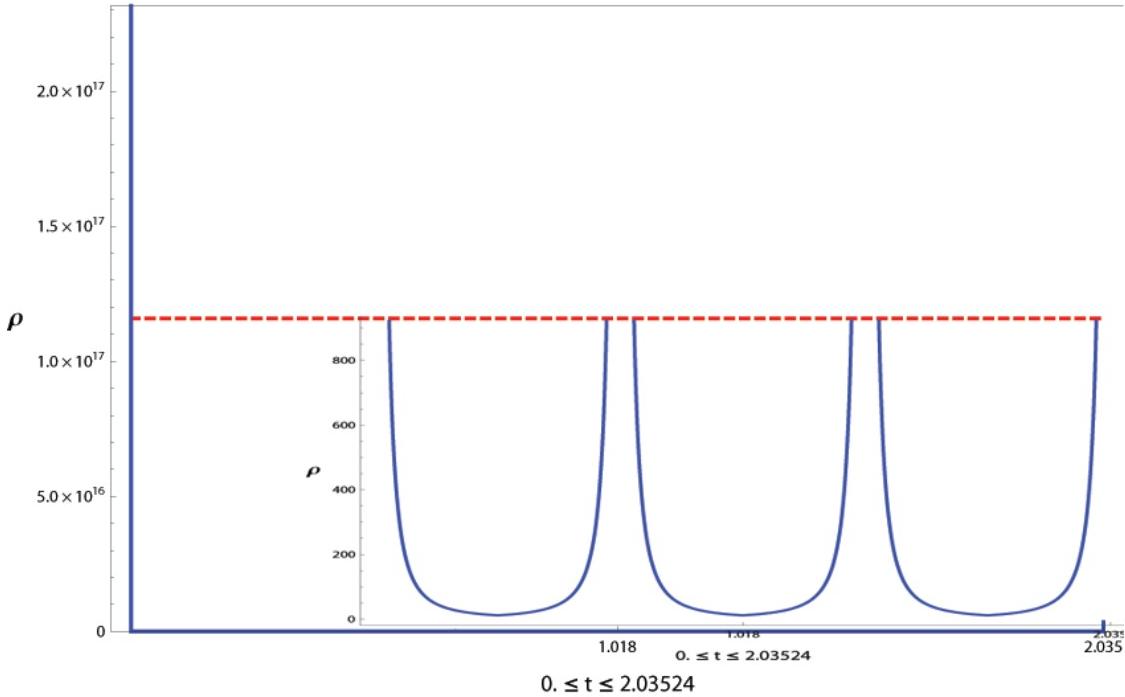
Fig. 5 Full problem: Density $\varrho(t)$. The inset shows periodic peaks in ϱ that reach the dotted red line and decrease to $\approx \varrho(0) \sqrt{\mathrm{sech}[F(0)]} \approx 11.6$. [Color online]

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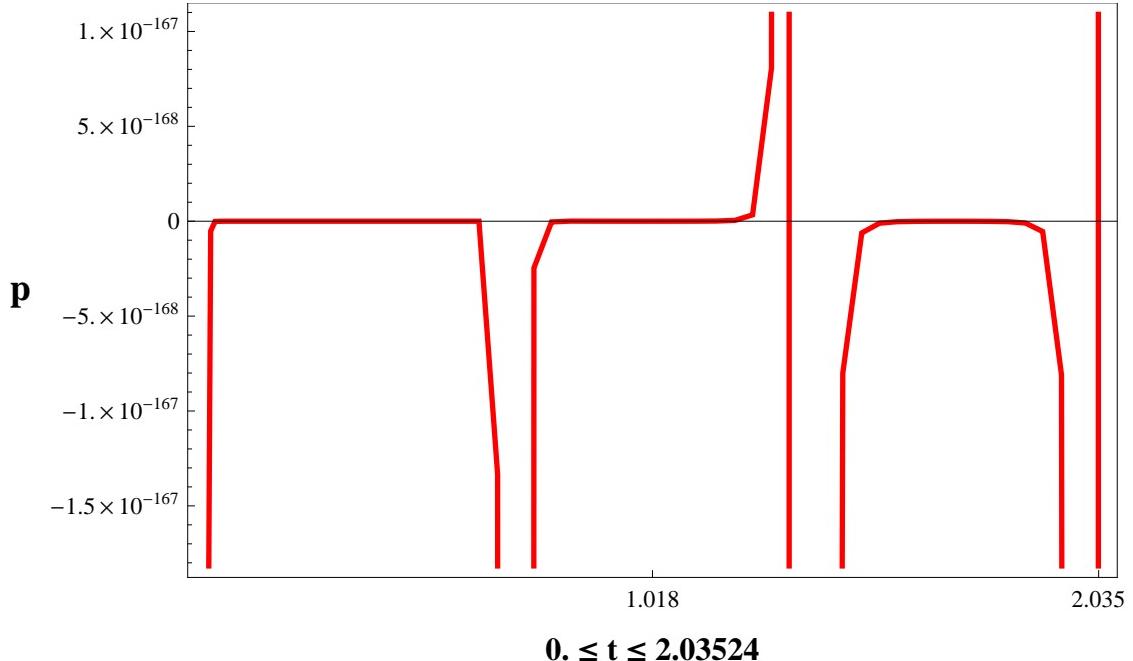
Fig. 6 Full problem: Pressure $p(t)$. Here $p(0) = 0$. [Color online]

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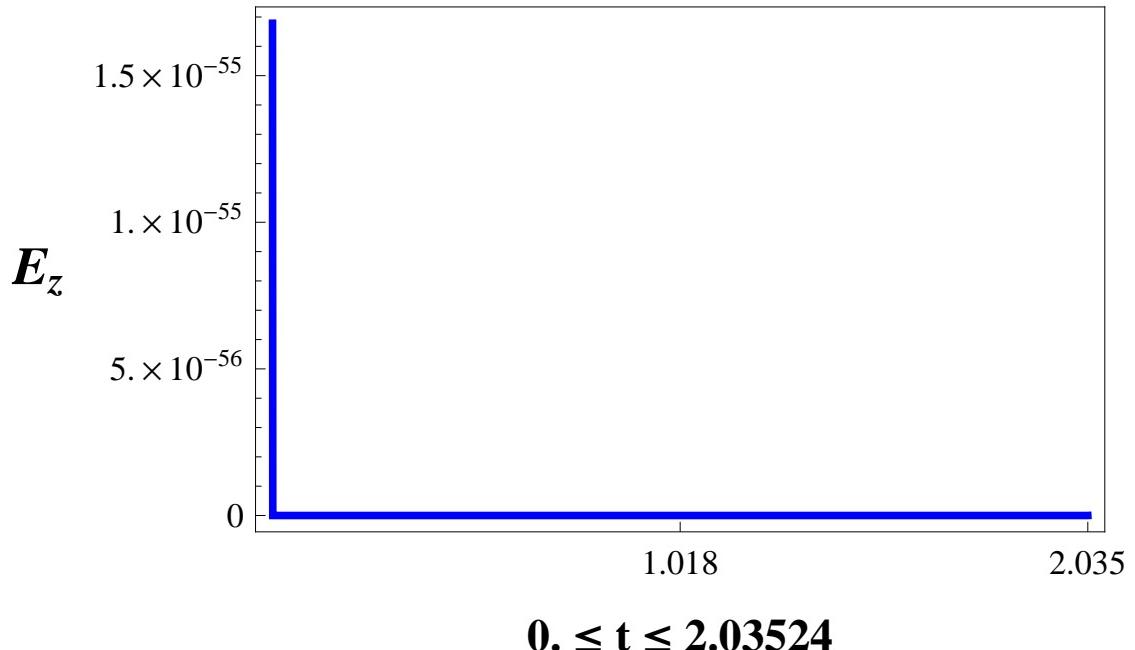
Fig. 7 Full problem: $E_z(t)$. [Color online]

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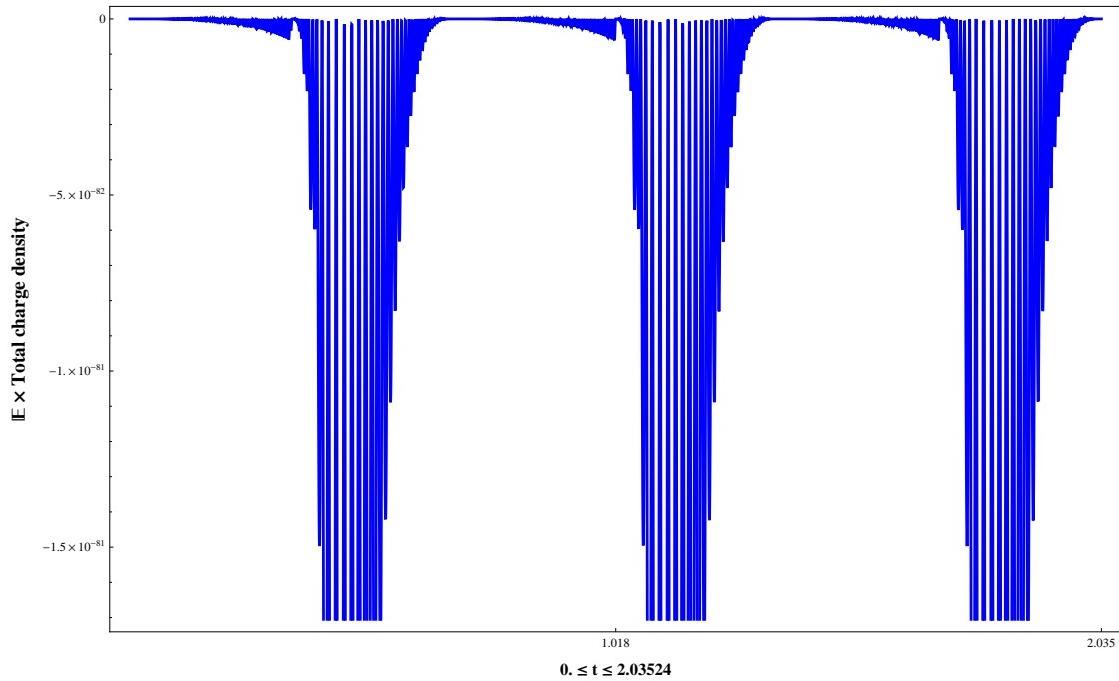
Fig. 8 Full problem: $a(t)^3 \times$ (total charge density). [Color online]

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